ERRATUM TO THE PAPER "BREUIL-KISIN-FARGUES MODULES WITH COMPLEX MULTIPLICATION"

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As noted in [2, Remark 1.2.2] the statement of [1, Lemma 3.25] is false. A counterexample is presented in [2, Example 4.3.4]. In this erratum we present this counterexample, discuss the failure of [1, Lemma 3.25] and its effects on the results of [1]. We thank Sean Howe for informing us about the error in [1, Lemma 3.25].

We use the notation from [1, Section 3], i.e., $C/\mathbb{Q}_p$ is a non-archimedean, algebraically closed field, A_{inf} Fontaine's period ring for $\mathcal{O}_C$, and $\epsilon = (1, \underset{\neq 1}{\zeta_p}, \ldots) \in C^\flat$, $\mu = [\epsilon] - 1$, $\tilde{\xi} := \frac{\varphi(\mu)}{\mu}$, $t = \log([\epsilon])$.

Example 0.1 ([1, Example 3.3])**.** For $d \in \mathbb{Z}$, the pair $A_{\mathrm{inf}}\{d\} := \mu^{-d} A_{\mathrm{inf}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(d)$ with Frobenius $\varphi_{A_{\mathrm{inf}}\{d\}} = \tilde{\xi}^d \varphi_{A_{\mathrm{inf}}}$ is a Breuil-Kisin-Fargues module, and in fact each Breuil-Kisin-Fargues module of rank 1 is isomorphic to some $A_{\mathrm{inf}}\{d\}$ ([1, Lemma 3.12]). The corresponding B_{dR}^+-latticed $\mathbb{Q}_p$-vector space (in the terminology of [2, Definition 4.2.1]) is $(\mathbb{Q}_p, t^{-d} B_{\mathrm{dR}}^+)$. Each $A_{\mathrm{inf}}\{d\}$ admits a canonical rigidification because $\tilde{x} = u \cdot p$ in A_{crys} for some unit (alternatively one can use [1, Lemma 4.3]).

According to [1, Lemma 3.28]

$$\mathrm{Ext}^1_{\mathrm{BKF}^{\circ}_{\mathrm{rig}}}(A_{\mathrm{inf}}, A_{\mathrm{inf}}\{d\}) \cong B_{\mathrm{dR}}/t^d B_{\mathrm{dR}}^+.$$

Now, a counterexample to [1, Lemma 3.25] will be provided by the case $d = 0$ with extension corresponding to $1/t$. Explicitly the corresponding extension of B_{dR}^+-latticed $\mathbb{Q}_p$-vector spaces is given by

$$0 \to (\mathbb{Q}_p \cdot e_1, B_{\mathrm{dR}}^+ \cdot e_1) \to (\mathbb{Q}_p \cdot e_1 \oplus \mathbb{Q}_p \cdot e_2, B_{\mathrm{dR}}^+ \cdot e_1 \oplus B_{\mathrm{dR}}^+ (\frac{1}{t} \cdot e_1 + e_2)) \to (\mathbb{Q}_p \cdot e_2, B_{\mathrm{dR}}^+ \cdot e_2) \to 0$$

as presented in [2, Example 3.1.4]. Now, the fiber functor $\omega_{\acute{e}t} \otimes C$ in [1, Lemma 3.25] from rigidifed Breuil-Kisin-Fargues modules to C-vector spaces factors over the functor to B_{dR}^+-latticed $\mathbb{Q}_p$-vector spaces, and this functor is not exact as a *filtered* functor as noted in [2, Example 3.1.4]: The above exact sequence maps in gr^0 to

$$0 \to C \to 0 \to C \to 0.$$

Indeed, the lattice $B_{\mathrm{dR}}^+ e_1 \oplus B_{\mathrm{dR}}^+(\frac{1}{t} \cdot e_1 + e_2)$ induces on $V_C := C \cdot e_1 \oplus C \cdot e_2$ the filtration

$$0 \subseteq \mathrm{Fil}^1 = C \cdot e_1 \subseteq \mathrm{Fil}^0 = V_C.$$

This example shows that the mistake in the "proof" of [1, 3.25] lies in the last five lines: Even though the element $v \otimes 1$ is part of some basis (e.g., $v \otimes 1 = e_1$ in the above example), it need not be part of an adapted basis. As far as I can tell this is the only mistake made.

We now discuss the effect of this mistake to the rest of the paper.

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(1) In [1, Section 2] we fix a filtered fiber functor $\omega_0 \otimes C \colon \mathcal{T} \to \mathrm{Vec}_C$ stating that later we can apply the discussion to rigidified Breuil-Kisin-Fargues modules. This is not true, however, restricting to CM rigidified Breuil-Kisin-Fargues modules the fiber functor $\omega_{ét}$ with its functorial filtration over C is a *filtered* fiber functor. Indeed, any fiber functor on a semisimple Tannakian category, which is equipped with a functorial filtration compatible with tensor products is necessary a filtered fiber functor as each exact sequence splits. Hence, the general theory of this section can be applied on the full Tannakian subcategory of CM-objects. We note that the type of a CM-object ([1, Definition 2.9]) only requires a functorial filtration on a fiber functor compatible with tensor products (and in characteristic 0 this data will automatically yield a filtered fiber functor on the CM-objects as explained above).

(2) The proof of [1, Lemma 3.27] cites [1, Lemma 3.25], however the claimed exactness is not used in the argument. Indeed, the claimed triviality of the filtration follows by correct compatibility of the filtration with tensor products. A similar argument occurs in [2, Theorem 4.3.5].

(3) With the above adjustements, the results in [1, Section 4, Section 5] are not affected.

References

[1] Johannes Anschütz. Breuil–Kisin–Fargues modules with complex multiplication. *Journal of the Institute of Mathematics of Jussieu*, 20(6):1855–1904, 2021.

[2] Sean Howe and Christian Klevdal. Admissible pairs and p-adic hodge structures i: Transcendence of the de rham lattice, 2023.

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