LECTURE NOTES ON TOPOI AND CONDENSED MATH

JOHANNES ANSCHÜTZ

Abstract. These notes are prepared for the M2 course "Topoi and condensed math" given at Paris-Saclay in 2024-2025.

Contents

1. Prerequisites and useful references

Prerequisites for the course are:

(1) algebra (rings/groups/modules/...)
(2) topology (compact Hausdorff spaces/...)
(3) category theory (category/functor/natural transformation/(co)limits/...)
(4) later: homological algebra (complexes/injectives/projectives/derived functor/...)

Knowledge of sheaves and sheaf cohomology will be helpful, scheme theory will only show up in some examples.

Useful references for this course are [Sch],[Sch19], [CS23], [RC24], [Ked24] (for condensed mathematics), and [AGV72], [Ans23] (for topoi). Another valuable reference is provided by [LS].

2. Introduction

This class has the following two aims:

(1) Give an introduction to condensed mathematics, and in particular, how to replace (properties of) topological spaces by (properties of) condensed sets,
(2) Explain the formal aspects of condensed mathematics in their natural framework, that is, topos theory.

According to [Sch, Introduction] the basic question adressed by condensed mathematics is the following:

How to do algebra when rings/modules/groups carry a topology?

In fact, algebraic structures like rings/groups/modules differ in an important aspect from topological spaces: given a homomorphism $f\colon Y \to X$ of rings/groups/modules, which is bijective, then the inverse f^{-1} is again a homomorphism of rings/groups/modules. The analog is not true for topological spaces, i.e., a continuous bijection $f\colon Y \to X$ of topological spaces is not necessarily a homeomorphism as the inverse f^{-1} is not necessarily continuous.

In a similar vein, a continuous bijection of topological rings/groups/modules need not be an isomorphism of topological rings/groups/modules. For example, this is the case for the identity map

$$\alpha\colon \mathbb{R}^{\mathrm{disc}} = (\mathbb{R},\ \text{discrete topology}\ ) \to \mathbb{R} = (\mathbb{R},\ \text{natural topology}). \tag{1}$$

This descrepancy leads to heavy foundational problems when doing homological algebra with topological modules/abelian groups/... . Namely, consider an exact sequence

$$M \xrightarrow{f} N \xrightarrow{g} Q$$

of continuous group homomorphisms of topological abelian groups, i.e., $H := \mathrm{Ker}(g) = \mathrm{Im}(f)$. Then the abelian group H can be given *two in general different* topologies:

(1) let H' be H equipped with the subspace topology for $\mathrm{Ker}(g) \subseteq N$,
(2) let H'' be H equipped with the quotient topology for the surjection $M \to H$.

There exists then a natural continuous bijection $H'' \to H'$, which need not be an isomorphism, e.g., if $f = \alpha$ from (1) and $g = 0$.

The solution to this problem, which is offered by condensed mathematics, rests on the following well-known fact.

Lemma 2.1. *Let $f\colon Y \to X$ be a continuous surjection of* compact Hausdorff spaces. *Then f is a topological quotient map. In particular, if f is bijective, then f^{-1} is continuous and f a homeomorphism.*

Proof. It suffices to show that $f(K) \subseteq X$ is closed for each closed subset $K \subseteq Y$. But K is compact (being closed in Y), and hence the image $f(K) \subseteq X$ is compact. But a compact set in the Hausdorff topological space X is closed. This implies that $f(K)$ is closed as desired. ☐

In condensed mathematics we will now profit from this fact by considering the category of *condensed sets*, which consists only of objects, which are "built out of compact Hausdorff spaces" in a sense that we specify below. Up to some set-theoretic issues, the category of condensed sets will be given by a *topos*, and thus be a category, which behaves very much like the category of sets. Let us specify what we mean by this.

Definition 2.2. (1) If $\mathcal{C}$ is a small[1] category, we define the category $\mathrm{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$ as the category of functors $\mathcal{C}^{\mathrm{op}} \to (\mathrm{Sets})$ (with morphisms given by natural transformations).
(2) A Grothendieck topology on a category $\mathcal{C}$ is a collection τ of sets of morphisms $\{Y_i \to Y\}_{i\in I}$ for each $Y \in \mathcal{C}$ (called coverings in τ, or just coverings) such that the following properties are satisfied:
  (a) If $Z \to Y$ is an isomorphism, then $\{Z \to Y\}_{I=\{*\}}$ is a covering.
  (b) If $\{Y_i \to Y\}_{i\in I}$ is a covering and $\{Z_{i,j} \to Y_i\}_{j\in J_i}$ is a covering for each $i \in I$, then $\{Z_{i,j} \to Y\}_{i\in I, j\in J_i}$ is a covering.
  (c) If $\{Y_i \to Y\}_{i\in I}$ and $Z \to Y$ is a morphism, then the fiber product $Y_i \times_Y Z$ exists for any $i \in I$ and $\{Y_i \times_Y Z \to Z\}_{i\in I}$ is a covering, where $Y_i \times_Y Z \to Z$ is the projection for each $i \in I$.

[1] We say that a category is small if the class of isomorphism classes of its objects is a set. This condition guarantees that $\mathrm{PSh}(\mathcal{C})$ is a category as the collection of natural transformations between two functors is actually a set.

(3) A presheaf $\mathcal{F} \in \mathrm{PSh}(\mathcal{C})$ is called a sheaf for a Grothendieck topology τ if for any covering $\{f_i \colon Y_i \to Y\}_{i\in I}$ in τ the natural map

$$\mathcal{F}(Y) \to \{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(Y_i) \mid p_{ij}^*(s_i) = q_{ij}^*(s_j) \text{ for all } i,j \in I\},\ \ s \mapsto f_i^*(s)$$

is bijective. Here, $p_{ij} \colon Y_i \times_Y Y_j \to Y_i$ and $q_{ij} \colon Y_i \times_Y Y_j \to Y_j$ are the two projections. A morphism of sheaves is a natural transformation of functors.

(4) A site is a pair of a small category $\mathcal{C}$ and a a Grothendieck topology τ on $\mathcal{C}$.

(5) A topos is a category, which is equivalent to the category $\mathrm{Sh}_\tau(\mathcal{C})$ of sheaves on some site $(\mathcal{C}, \tau)$.

For example if Z is a topological space, then its category of sheaves of sets is a topos. Let us specify some of the good categorical properties of topoi, that we will prove during the course.

Theorem 2.3 (Theorem 3.11)**.** *Let $\mathfrak{X}$ be a topos.*

(1) *$\mathfrak{X}$ has all colimits and all limits.*
(2) *Filtered*[2] *colimits in $\mathfrak{X}$ commute with finite limits.*
(3) *A morphism in $\mathfrak{X}$ is an isomorphism if and only if it is a monomorphism and an epimorphism.*

Moreover, in $\mathfrak{X}$ colimits are universal and epimorphisms are effective (we refer to Theorem 3.11 for explanations of these terms).

Because of the similarity with the category of sets, one often transfers set-theoretic notions to a topos: a morphism $f \colon X \to Y$ in a topos is

- injective if it is a monomorphism.
- surjective if it is an epimorphism.

In loose terms, a topos $\mathfrak{X} = \mathrm{Sh}(\mathcal{C})$ is "built out" of the objects in $\mathcal{C}$ under colimits. As said before, condensed sets will be built out of compact Hausdorff spaces. In fact, we will even built them from the very important class of profinite sets.

Definition 2.4. A profinite set is a compact Hausdorff space, which is totally disconnected, i.e., the only non-empty connected subsets are singletons.

We will study profinite sets in more detail. In particular, we will show the following theorem.

Theorem 2.5 (Theorem 3.35)**.** (1) *A topological space S is a profinite set if and only if $S \cong \varprojlim_{i\in I} S_i$ is a cofiltered*[3] *limit of finite sets S_i.*
(2) *Each compact Hausdorff space admits a surjection from a profinite set.*
(3) *The full subcategory $\mathrm{Prof} \subseteq \mathrm{Top} := \{$topological spaces$\}$ of profinite sets is stable under all limits.*

Profinite sets exist therefore in abundance, e.g., the Cantor set $\{0,1\}^{\mathbb{N}}$ is profinite, or the p-adic integers $\mathbb{Z}_p = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n$ for a prime p.

There are set-theoretic issues when working with the category of profinite sets due to the fact that it is a proper class. For this reason we will cut them down in size. We fix an infinite cardinal κ, e.g., $\kappa = \omega_1$ is the first uncountable ordinal.

Definition 2.6. A profinite set S is called κ-light, or just a κ-profinite set, if the set $\{U \subseteq S \mid U \text{ open and closed}\}$ has cardinality $< \kappa$. We let $\mathrm{Prof}_\kappa \subseteq \mathrm{Prof}$ be the full subcategory of κ-profinite sets.

It follows from Theorem 2.5 that automatically $|S| < 2^\kappa$, so that Prof_κ is a *small* category. We can now define κ-condensed sets for an uncountable[4] cardinal κ.

Definition 2.7. (1) Define a Grothendieck topology on Prof_κ by saying that a family $\{S_i \to S\}_{i\in I}$ is a covering if I is finite and $\coprod_{i\in I} S_i \to S$ is surjective.
(2) A κ-condensed set is a sheaf on Prof_κ for this Grothendieck topology.
(3) We let $\mathrm{CondSet}_\kappa := \mathrm{Sh}(\mathrm{Prof}_\kappa)$ be the topos of κ-condensed sets.

We can conclude that $\mathrm{CondSet}_\kappa$ has the excellent categorical properties mentioned in Theorem 2.3. To lighten notation, we will later often omit the κ if it is understood from the context. The most important example of a condensed set is the following.

---

[2]We recall that a colimit is filtered if its indexing category I is fitered, and this means that I is non-empty, for $i, j \in I$ there exists k in I with morphisms $i \to k,\ j \to k$, and for two morphisms $i \rightrightarrows j$ there exists a morphism $j \to k$ such that the two compositions $i \rightrightarrows j \to k$ agree.

[3]A cofiltered limit is a limit over a cofiltered category, and a category is cofiltered if its opposite is filtered.

[4]If κ is countable, then Prof_κ is just the category of finite sets.

Lemma 2.8. *Let X be a topological space. Then the functor $\underline{X}\colon \mathrm{Prof}_\kappa^{\mathrm{op}} \to \mathrm{Sets}$, $S \mapsto \mathrm{Hom}_{\mathrm{cts}}(S, X)$ is a κ-condensed set.*

Proof. Let $\{f_i\colon S_i \to S\}_{i\in I}$ be a covering in Prof_κ. We need to see the map

$$\Phi\colon \underline{X}(S) \to \{(g_i)_{i\in I} \in \prod_{i\in I} \underline{X}(S_i) \mid g_i \circ p_{ij} = g_j \circ q_{ij} \text{ for all } i,j \in I\},\ g \mapsto g \circ f_i$$

is bijective, where $p_{ij}\colon S_i \times_S S_j \to S_i$, $q_{ij}\colon S_i \times_S S_j \to S_j$ are the projections. Because the morphisms $f_i\colon S_i \to S$ are jointly surjective, i.e., $S = \cup_{i\in I} f_i(S_i)$, the map Φ is injective. Assume that $(g_i)_{i\in I} \in \prod\limits_{i\in I} S_i$ is an element of the target. We define a map $g \in S \to X$ as follows: if $s \in S$, choose some $i \in I$, and $t \in S_i$ such that $f_i(t) = s$. Now set $g(s) := g_i(t)$. We need to see that this map is well-defined: take $t' \in S_j$ with $f_j(t') = s$. Then $(t, t') \in S_i \times_S S_j$ by the definition of the fiber product, and

$$g_i(t) = g_i \circ p_{ij}(t, t') = g_j \circ q_{ij}(t, t') = g_j(t')$$

as desired. Hence, $g\colon S \to X$ is a well-defined map, and g clearly satisfies that $g \circ f_i = g_i$ for $i \in I$. Now, $\coprod\limits_{i\in I} S_i \to S$ is a surjection of compact Hausdorff spaces (we use here that I is finite!), and thus a quotient map by Lemma 2.1. Because each g_i is continuous, we can therefore conclude that g is continuous, i.e., $g \in \underline{X}(S)$. □

We will prove the following properties of the functor $\underline{(-)}$ (among others).

Theorem 2.9 (Lemma 4.10, Lemma 4.24, Lemma 4.3, Lemma 4.18)**.**

(1) *The functor $\underline{(-)}\colon \mathrm{Top} \to \mathrm{CondSet}_\kappa$ commutes with limits, and sends proper/open surjections of locally compact Hausdorff spaces (satisfying a mild set-theoretic condition) to epimorphisms.*
(2) *The functor $\underline{(-)}$ admits a left adjoint $\mathrm{CondSet}_\kappa \to \mathrm{Top}$, $X \mapsto X(*)_{\mathrm{top}}$, and $\underline{(-)}$ is fully faithful on κ-compactly generated topological spaces.*
(3) *The functor $\underline{(-)}$ induces an equivalence between compact Hausdorff spaces, admitting a surjection by a κ-profinite set, and quasi-compact, quasi-separated κ-condensed sets.*

Here, a topological space Z is κ-compactly generated if a map $f\colon Z \to Y$ to a topological space is continuous if and only if the composition $S \to Z \to Y$ is continuous for any κ-profinite set S. For example, each first countable topological space Z, e.g., Z metrizable, is ω_1-compactly generated because f is continuous if and only if it f maps convergent sequences to convergent sequences. However, convergent sequences in Z are exactly maps from the one-point-compactification $\mathbb{N} \cup \{\infty\}$ of $\mathbb{N}$ to Z. Thus Theorem 2.9 shows that the functor $\underline{(-)}$ preserves a lot of informations on topological spaces.

Replacing functors with values in sets by functors with values in abelian groups, one obtains the category CondAb_κ of κ-condensed abelian groups, which allows to mix homological algebra with topological abelian groups. More generally, given a topos $\mathfrak{X}$, we can consider its abelian group objects $\mathrm{Ab}(\mathfrak{X})$, i.e., sheaves of abelian groups.

The category $\mathrm{Ab}(\mathfrak{X})$ behaves very well.

Theorem 2.10.

(1) *The category $\mathrm{Ab}(\mathfrak{X})$ is a Grothendieck abelian category, i.e., it is abelian, filtered colimits are exact and it has a generator.*
(2) *Assume $\mathfrak{X} = \mathrm{CondSet}_\kappa$. If κ is a strong limit cardinal, i.e., $\lambda < \kappa$ implies $2^\lambda < \kappa$, then products are exact in $\mathrm{Ab}(\mathfrak{X})$. For general κ, countable products are exact.*

We will discuss properties of Grothendieck abelian categories in more detail later. For now we just note that Theorem 2.10 implies that $\mathrm{CondAb}_\kappa = \mathrm{Ab}(\mathrm{CondSet}_\kappa)$ behaves very much like the category of abelian groups, especially if κ is a strong limit cardinal.

It is instructive to revisit (1) in condensed sets: the morphism $\underline{\alpha}\colon \underline{\mathbb{R}^{\mathrm{disc}}} \to \underline{\mathbb{R}}$ of condensed sets is injective, but not surjective. Indeed, given $S \in \mathrm{Prof}_\kappa$, then

$$\underline{\mathbb{R}^{\mathrm{disc}}}(S) = C^{\mathrm{lc}}(S, \mathbb{R}) := \{\varphi\colon S \to \mathbb{R} \mid \varphi \text{ locally constant}\}$$

while

$$\underline{\mathbb{R}}(S) = C(S, \mathbb{R}) := \{\varphi\colon S \to \mathbb{R} \mid \varphi \text{ continuous}\}.$$

If S is finite, then $C^{\mathrm{lc}}(S, \mathbb{R}) = C(S, \mathbb{R})$, but both differ in general, e.g., if $S = \mathbb{N} \cup \{\infty\}$. Hence, the cokernel of α ought to be the functor $S \mapsto C(S, \mathbb{R})/C^{\mathrm{lc}}(S, \mathbb{R})$. That this is indeed true requires some work. In fact, we will aim for a more general statement.

Theorem 2.11. *For any κ-compactly generated compact Hausdorff space K, and any abelian group M (viewed as a discrete topological space), the condensed cohomology $H^*_{\mathrm{CondSet}_\kappa}(\underline{K}, \underline{M})$ is naturally isomorphic to the sheaf cohomology on K with coefficients in M.*

Here, the condensed cohomology of $\underline{K}$ is a special case of the topos-theoretic notion of cohomology: given a topos $\mathfrak{X}$ and an object X, then $H^*_{\mathfrak{X}}(X, -)$ denotes the right derived functors of the functor

$$\mathrm{Ab}(\mathfrak{X}) \to \mathrm{Ab},\ \mathcal{F} \mapsto \mathcal{F}(X) = \mathrm{Hom}_{\mathfrak{X}}(X, \mathcal{F}).$$

In particular, if K is moreover locally contractible, then sheaf cohomology on K agrees with singular cohomology of K, and hence is quite explicit. Theorem 2.11 is vital for calculations in CondAb_κ. We will illustrate this by proving the following non-obvious result.

Theorem 2.12. *We have*

$$\mathrm{Ext}^i_{\mathrm{CondAb}_\kappa}(\underline{S^1}, \underline{\mathbb{Z}}) \cong \begin{cases} 0 & \textit{if } i \neq 1 \\ \mathbb{Z} & \textit{if } i = 1, \end{cases}$$

with the generator in degree 1 *given by the extension* $0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{R}} \to \underline{S^1} \to 0$.

3. Condensed sets

We now start to develop enough topos theory to be able to prove Theorem 2.3. As defined in Definition 2.2 a general topos is the category of sheaves on some site.

Topos theory is very general, and we will provide some examples later, which show their usefulness in other contexts, e.g., étale cohomology or group cohomology.

3.1. Interlude: sites, sheaves and topoi.

We recall the definition of a site and a topos, Definition 2.2.

Definition 3.1. (1) If $\mathcal{C}$ is a small[5] category, we define the category $\mathrm{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$ as the category of functors $\mathcal{C}^{\mathrm{op}} \to (\mathrm{Sets})$ (with morphisms given by natural transformations).

(2) A Grothendieck topology on a category $\mathcal{C}$ is a collection τ of sets of morphisms $\{Y_i \to Y\}_{i\in I}$ for each $Y \in \mathcal{C}$ (called coverings in τ, or just coverings) such that the following properties are satisfied:
   (a) If $Z \to Y$ is an isomorphism, then $\{Z \to Y\}_{I=\{*\}}$ is a covering.
   (b) If $\{Y_i \to Y\}_{i\in I}$ is a covering and $\{Z_{i,j} \to Y_i\}_{j\in J_i}$ is a covering for each $i \in I$, then $\{Z_{i,j} \to Y\}_{i\in I, j\in J_i}$ is a covering.
   (c) If $\{Y_i \to Y\}_{i\in I}$ and $Z \to Y$ is a morphism, then the fiber product $Y_i \times_Y Z$ exists for any $i \in I$ and $\{Y_i \times_Y Z \to Z\}_{i\in I}$ is a covering, where $Y_i \times_Y Z \to Z$ is the projection for each $i \in I$.

(3) A presheaf $\mathcal{F} \in \mathrm{PSh}(\mathcal{C})$ is called a sheaf for a Grothendieck topology τ if for any covering $\{f_i \colon Y_i \to Y\}_{i\in I}$ in τ the natural map
$$\mathcal{F}(Y) \to \{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(Y_i) \mid p_{ij}^*(s_i) = q_{ij}^*(s_j) \text{ for all } i,j \in I\},\ s \mapsto f_i^*(s)$$
is bijective. Here, $p_{ij} \colon Y_i \times_Y Y_j \to Y_i$ and $q_{ij} \colon Y_i \times_Y Y_j \to Y_j$ are the two projections. A morphism of sheaves is a natural transformation of functors.

(4) A site is a pair of a small category $\mathcal{C}$ and a a Grothendieck topology τ on $\mathcal{C}$.

(5) A topos is a category, which is equivalent to the category $\mathrm{Sh}_\tau(\mathcal{C})$ of sheaves on some site $(\mathcal{C}, \tau)$.

Given a functor $\mathcal{F} \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$, $Y \mapsto \mathcal{F}(Y)$ and a morphism $Z \to Y$ in $\mathcal{C}$, then we use the notation $\mathcal{F}(Y) \to \mathcal{F}(Z)$, $s \mapsto s_{|Z}$, and call it the "restriction" from Y to Z. We also use the notation $\Gamma(Y, \mathcal{F}) := \mathcal{F}(Y)$.

We can directly give relevant examples.

Example 3.2. (1) Let Z be a topological space, and let $\mathrm{Ouv(Z)} := \{\mathrm{U} \subseteq \mathrm{Z}\ \mathrm{open}\}$ be its poset of open sets (viewed as a category). Now we say that collection $\{U_i \to U\}_{i\in I}$, i.e., a collection of open subsets $U_i \to U$, is a covering if for each $u \in U$ there exists some $i \in I$ with $u \in U_i$. It is easily verified that this defines a Grothendieck topology. The sheaves for this Grothendieck topology are exactly the sheaves on Z (as introduced in any discussion of sheaves on topological spaces).

(2) Let $\mathcal{C} = \mathrm{CHaus}$ be the category of compact Hausdorff spaces. Given $K \in \mathrm{CHaus}$ and a collection $\{K_i \to K\}_{i\in I}$ of morphisms in CHaus, we say that it is a covering of K if I is finite and the morphism $\coprod_{i\in J} K_i \to K$ is surjective. This notion of coverings is easily seen to define a Grothendieck topology. Restricting to κ-profinite sets recovers Definition 2.7.

(3) Given any small category $\mathcal{C}$, we can say that a covering $\{Y_i \to Y\}_{i\in I}$ is a collection such that I contains a single element $i \in I$, and $Y_i \to Y$ is an isomorphism. For this Grothendieck topology, the category of sheaves is just $\mathrm{PSh}(\mathcal{C})$ as the sheaf condition becomes vacuous.

(4) One of the central motivations for introducing topoi was étale cohomology: let X be a scheme, and set $X_{\text{ét}} := \{Y \to X \text{ étale}\}$ with coverings $\{f_i \colon Y_i \to Y\}_{i\in I}$ given by collections of maps, such that $\bigcup_{i\in I} f_i(Y_i) = Y$. The cohomology of the topos $\mathrm{Sh}(X_{\text{ét}})$, i.e., the derived functor of global sections, calculates the étale cohomology of X.

(5) Given a group G, we can consider $\mathcal{C}$ as the category of G-sets, i.e., the category of sets with a G-action and G-equivariant morphisms. We define a collection $\{S_i \to S\}_{i\in I}$ of morphisms in $\mathcal{C}$ to be a covering if $\coprod_{i\in I} S_i \to S$ is surjective. The cohomology of this topos calculates the group cohomology of G.

Our first task will be to understand limits and colimits in a topos. The case of limits is easy.

[5]We say that a category is small if the class of isomorphism classes of its objects is a set. This condition guarantees that $\mathrm{PSh}(\mathcal{C})$ is a category as the collection of natural transformations between two functors is actually a set.

Lemma 3.3. *Let $\mathfrak{X}$ be a topos. Then* X *has all limits, and these can be computed as follows: pick any site $(\mathcal{C}, \tau)$, such that* $\mathrm{X} \cong \mathrm{Sh}(\mathcal{C})$*. Then the fully faithful inclusion* $\mathrm{Sh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ *preserves limits, and limits in* $\mathrm{PSh}(\mathcal{C})$ *exists and are computed pointwise, i.e., for any $X \in \mathcal{C}$ the functor* $\mathrm{PSh}(\mathcal{C}) \to (\mathrm{Sets})$, $\mathcal{F} \mapsto \mathcal{F}(X)$ *commutes with limits.*

Proof. We leave the description of limits in $\mathrm{PSh}(\mathcal{C})$ as an exercise. For limits of sheaves, it suffices to see that the full subcategory $\mathrm{Sh}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ is stable under limits. This follows because the sheaf condition is defined in terms of limits and limits commute with limits. □

The case of colimits needs more works: we need to discuss sheafification first.

Definition 3.4. Let $\mathcal{C}$ be any category. Let $Y \in \mathcal{C}$. We say that a collection $\mathcal{U} = \{f_i \colon Y_i \to Y\}_{i\in I}$ of morphisms in $\mathcal{C}$ refines another collection $\mathcal{V} = \{g_j \colon Z_j \to Y\}_{j\in J}$ if there exists a map $\varphi \colon I \to J$, and morphisms $h_i \colon Y_i \to Z_{\varphi(i)}, i \in I$, over Y, i.e., $g_{\varphi(i)} \circ h_i = f_i$ for $i \in I$.

Let $(\mathcal{C}, \tau)$ be a site. Given a covering $\mathcal{U} = \{Y_i \to Y\}_{i\in I}$ of $Y \in \mathcal{C}$, and a presheaf $\mathcal{F} \in \mathrm{PSh}(\mathcal{C})$, we set

$$\Gamma(\mathcal{U}, \mathcal{F}) := \mathrm{eq}(\prod_{i\in I} \mathcal{F}(Y_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(Y_i \times_Y Y_j)),$$

where the two morphisms in the equalizer are induced by the two projections $Y_i \times_Y Y_j \to Y_i$, $Y_i \times_Y Y_j \to Y_j$. Note that the sheaf condition for $\mathcal{F}$ is asking that the natural map $\Gamma(Y, \mathcal{F}) \to \Gamma(\mathcal{U}, \mathcal{F})$ is an isomorphism for any covering $\mathcal{U}$.

Lemma 3.5. *Assume that the covering $\mathcal{U} = \{Y_i \to Y\}_{i\in I}$ refines the cover $\mathcal{V} = \{Z_j \to Y\}_{j\in J}$ as witnessed by a map $\varphi \colon I \to J$, and morphisms $g_i \colon Y_i \to Z_{\varphi(i)}, i \in I$, over Y. Let $\mathcal{F} \in \mathrm{PSh}(\mathcal{C})$.*

(1) *The map*
$$r_{\mathcal{U},\mathcal{V}} \colon \Gamma(\mathcal{V}, \mathcal{F}) \to \Gamma(\mathcal{U}, \mathcal{F}),\ (s_j)_{j\in J} \mapsto (g_i^*(s_{\varphi(i)}))_{i\in I}$$
is well defined and independent of the choice of $\varphi, g_i, i \in I$.

(2) *If $\mathcal{F}$ is separated, i.e., for any covering $\mathcal{U}'$ of an object Z the morphism $\Gamma(Z, \mathcal{F}) := \mathcal{F}(Z) \to \Gamma(\mathcal{U}, \mathcal{F})$ is injective, and the natural map $\mathcal{F}(Y) \to \Gamma(\mathcal{U}, \mathcal{F})$ is bijective, the same holds for $\mathcal{F}(Y) \to \Gamma(\mathcal{V}, \mathcal{F})$.*

Proof. The well-definedness follows easily by using the map $(g_{i_1}, g_{i_2}) \colon Y_{i_1} \times_Y Y_{i_2} \to Z_{\varphi(i_1)} \times_Y Z_{\varphi(i_2)}$.

Let now $j \in J$. Assume that $h_i \colon Y_i \to Z_j$ is any map over Y. Then we have a map $h := (g_i, h_i) \colon Y_i \to Z_{\varphi(i)} \times_Y Z_j$. Let $p_1 \colon Z_{\varphi(i)} \times_Y Z_j \to Z_{\varphi(i)}$, $p_2 \colon Z_{\varphi(i)} \times_Y Z_j \to Z_j$ be the projections. Then

$$g_i^*(s_{\varphi(i)}) = h^*(p_1^*(s_{(\varphi(i))})) \overset{(s_j)_j \in \Gamma(\mathcal{V},\mathcal{F})}{=} h^*(p_2^*(s_j)) = h_i^*(s_j).$$

This implies that $r_{\mathcal{U},\mathcal{V}}$ is independent of $\varphi, g_i, i \in I$.

Now, assume that the map $r_{\mathcal{U}} \colon \mathcal{F}(V) \to \Gamma(\mathcal{U}, \mathcal{F})$ is bijective. Then clearly,

$$r_{\mathcal{V}} \colon \mathcal{F}(V) \to \Gamma(\mathcal{V}, \mathcal{F})$$

is injective, as its composite with $r_{\mathcal{U},\mathcal{V}}$ is $r_{\mathcal{U}}$. Take now $(s_j)_j \in \Gamma(\mathcal{V}, \mathcal{F})$. Then $r_{\mathcal{U},\mathcal{V}}((s_j)_j) = r_{\mathcal{U}}(s)$ for some $s \in \mathcal{F}(Y)$. Now fix $j \in J$ and consider $f_j \colon Z_j \to Y$. We need to see that $s_{|Z_j} = f_j^*(s) = s_j$. The collection

$$\{h_{i,j} \colon Y_i \times_Y Z_j \to Z_j\}_{i\in I}$$

is a covering of Z_j. Let $g_{i,j} \colon Y_i \times_Y Z_j \to Y_i$ be the projection. Then for all $i \in I$

$$h_{i,j}^*(f_j^*(s)) = g_{i,j}^*(s_{|Y_i}) = g_{i,j}^*(g_i^*(s_{\varphi(i)}))$$

and if $g := (g_i, \mathrm{Id}_{Z_j}) \colon Y_i \times_Y Z_j \to Z_{\varphi(i)} \times_Y Z_j$ this equals

$$g^*(s_{\varphi(i))|Z_{\varphi(i)} \times_Y Z_j}) = g^*(s_{j|Z_{\varphi(i)} \times_Y Z_j}) = h_{i,j}^*(s_j)$$

by definition of $\Gamma(\mathcal{V}, \mathcal{F})$. If now $\mathcal{F}$ is separated, this implies that

$$f_j^*(s) = s_j$$

and thus the second statement is proven. □

Definition 3.6. Let $(\mathcal{C}, \tau)$ be a site. Let $\mathcal{F} \in \mathrm{PSh}(\mathcal{C})$, then we define the presheaf $\mathcal{F}^+ \in \mathrm{PSh}(\mathcal{C})$ as

$$(Y \in X_{\mathrm{t}}) \mapsto \mathcal{F}^+(Y) := \varinjlim_{\mathcal{U} \text{ covering of } Y} \Gamma(\mathcal{U}, \mathcal{F}),$$

where the colimit is taken along the maps $r_{\mathcal{U},\mathcal{V}}$ from Lemma 3.5 if $\mathcal{U}$ refines $\mathcal{V}$. If $f \colon Y \to Z$ is a morphism in $\mathcal{C}$ and $\mathcal{U} = \{Z_j \to Z\}_{j\in J}$ a covering of Z, then $\mathcal{U} \times_Z Y := \{Y \times_Z Z_j \to Y\}_{j\in J}$ is a covering of Y and the pullbacks along $Y \times_Z Z_j \to Z_j$ define a natural morphism

$$\Gamma(\mathcal{U}, \mathcal{F}) \to \Gamma(\mathcal{U} \times_Z Y, \mathcal{F}),$$

which is compatible with refinement. Passing to the colimit over all coverings yields the restriction morphism

$$\mathcal{F}^+(Z) \to \mathcal{F}^+(Y)$$

for $\mathcal{F}^+$.

Remark 3.7. The colimit $\varinjlim\limits_{\mathcal{U} \text{ covering of } Y}$ is *filtered.* Indeed, given two coverings $\{Y_i \to Y\}_{i\in I}$, $\{Z_j \to Y\}_{j\in J}$ are two coverings, then $\{Y_i \times_Y Z_j \to Y\}_{(i,j)\in I\times J}$ is a common refinement of both.

We can now establish the existence of sheafification.

Theorem 3.8. *Let $(\mathcal{C}, \tau)$ be a site.*

(1) *If $\mathcal{F} \in \mathrm{PSh}(\mathcal{C})$, then the presheaf $\mathcal{F}^+$ is separated.*
(2) *If $\mathcal{F} \in \mathrm{PSh}(\mathcal{C})$ is separated, then $\mathcal{F}^+$ is a sheaf and $\mathcal{F} \to \mathcal{F}^+$ is the initial morphism from $\mathcal{F}$ to a sheaf.*

In particular, the functor $\mathcal{F} \mapsto \mathcal{F}^\sharp := (\mathcal{F}^+)^+$ yields a left adjoint ("sheafification") to the inclusion $\mathrm{Sh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$. Finally, the sheafification $(-)^\sharp\colon \mathrm{PSh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$ is an exact functor, i.e., preserves finite limits.

Proof. The first statement follows easily from the definition of $\mathcal{F}^+$ and of being a separated presheaf. Let us prove the second and assume that $\mathcal{F}$ is a separated presheaf. We have to show that $\mathcal{F}^+$ is a sheaf. Thus fix $Y \in \mathcal{C}$ and a covering $\mathcal{U} = \{Y_i \to Y\}_{i\in I}$. We have to show that

$$\alpha\colon \mathcal{F}^+(Y) \to \Gamma(\mathcal{U}, \mathcal{F}^+)$$

is bijective. As $\mathcal{F}^+$ is separated (by (1)) the map α is injective. Let

$$(s_i)_{i\in I} \in \Gamma(\mathcal{U}, \mathcal{F}^+).$$

Then there exist coverings $\mathcal{U}_i = \{Z_{i,j} \to Y_i\}_{j\in J_i}$ such that $s_i \in \mathcal{F}^+(Y_i)$ can be represented by some $(t_{i,j})_{j\in J_i} \in \Gamma(\mathcal{U}_i, \mathcal{F})$. Now the collection of morphisms

$$\mathcal{V} = \{Z_{i,j} \to Y_i \to Y\}_{i\in I, j\in J_i}$$

is a covering of Y. By construction, the element

$$r_{\mathcal{V},\mathcal{U}}((s_i)_i) \in \Gamma(\mathcal{V}, \mathcal{F}^+)$$

lies in the subset $\Gamma(\mathcal{V}, \mathcal{F}^+) \cap \prod_{i,j} \mathcal{F}(Z_{i,j})$ (here we identify $\mathcal{F}(Z_{i,j}) \subseteq \mathcal{F}^+(Z_{i,j})$ as a subset by separatedness of $\mathcal{F}$). Now, if $\mathcal{G}$ is any separated presheaf and $\mathcal{W} = \{Z_k \to Z\}_{k\in K}$ one has the equality

$$\Gamma(\mathcal{W}, \mathcal{G}^+) \cap \prod_{k\in K} \mathcal{G}(Z_i) = \Gamma(\mathcal{W}, \mathcal{G})$$

because the map $\prod\limits_{k,l\in K} \mathcal{G}(Z_k \times_Z Z_l) \to \prod\limits_{k,l\in K} \mathcal{G}^+(Z_k \times_Z Z_l)$ is injective. Applied in our situation we see that

$$t := r_{\mathcal{U},\mathcal{V}}((s_i)_i) \in \Gamma(\mathcal{V}, \mathcal{F}) = \Gamma(\mathcal{V}, \mathcal{F}^+) \cap \prod_{i,j} \mathcal{F}(Z_{i,j}).$$

But this implies that $t \in \Gamma(\mathcal{V}, \mathcal{F}) \subseteq \mathcal{F}^+(Y)$ restricts to $(s_i)_i \in \Gamma(\mathcal{V}, \mathcal{F}^+)$ as desired. It follows from the definition that a presheaf $\mathcal{F}$ is sheaf if and only if the map $\mathcal{F} \to \mathcal{F}^\sharp$ is an isomorphism. This implies that $(-)^\sharp$ satisfies the universal property of sheafification. In particular, it is left adjoint to the inclusion $\mathrm{Sh}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$. It suffices to see that $(-)^\sharp$ commutes with finite limits. By Lemma 3.3 limits in $\mathrm{Sh}(\mathcal{C})$ agree with limits in $\mathrm{PSh}(\mathcal{C})$. As *filtered* colimits commute with finite limits, it follows by Remark 3.7 that the functor $(-)^+\colon \mathrm{PSh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ commutes with finite limits. In particular, $(-)^\sharp = ((-)^+)^+$ commutes with finite limits. □

Remark 3.9. Given a morphism $f\colon \mathcal{F} \to \mathcal{G}$ of sheaves, then it is an epimorphism if and only if for each object $Y \in \mathcal{C}$, and each section $s \in \mathcal{G}(Y)$ there exists a covering $\{Y_i \to Y\}_{i\in I}$ and elements $t_i \in \mathcal{F}(Y_i)$, $i \in I$, such that $s_{|Y_i} = f(t_i)$ for each $i \in I$. In particular, $\mathcal{F}(Y) \to \mathcal{G}(Y)$ need *not* be a surjective, even if f is an epimorphism.

Remark 3.10. Let Prof_κ be the site of κ-profinite sets from Definition 2.7, and let $*$ be a one point profinite set. Then each surjection $S \to *$ admits a splitting. But this implies that the covering $\{* \to *\}$ is cofinal among all coverings. From the description of sheafification we can conclude that $\mathcal{F}(*) = \mathcal{F}^\sharp(*)$ for any $\mathcal{F} \in \mathrm{PSh}(\mathrm{Prof}_\kappa)$.

We can now prove Theorem 2.3.

Theorem 3.11. *Let $\mathfrak{X}$ be a topos.*

(1) *$\mathfrak{X}$ has all colimits and all limits.*
(2) *Filtered colimits in $\mathfrak{X}$ commute with finite limits, i.e., are exact.*
(3) *A morphism in $\mathfrak{X}$ is an isomorphism if and only if it is a monomorphism and an epimorphism.*
(4) *For a morphism $Y \to X$ in $\mathfrak{X}$ the functor $\mathfrak{X}/X \to \mathfrak{X}/Y$, $(Z \to X) \mapsto (Y \times_X Z \to Y)$ preserves colimits. Here, $\mathfrak{X}/X$ is the category of objects $Z \to X$ with a morphism to X (and morphisms respecting the morphism to X).*
(5) *An epimorphism $f\colon Y \to X$ in $\mathfrak{X}$ is an effective epimorphism, i.e., the natural morphism* $\mathrm{coeq}(Y \times_X Y \rightrightarrows Y) \to X$ *is an isomorphism.*

Proof. That $\mathfrak{X}$ has all limits was noted in Lemma 3.3. The case of colimits follows formally from Theorem 3.8, and we can give the following recipe for computing them: let (C, τ) be a site with $\mathfrak{X} = \mathrm{Sh}(\mathcal{C})$. Given a diagram $\mathcal{F}_i, i \in I$, in $\mathrm{Sh}(\mathcal{C})$, then the pointwise colimit $\mathcal{G} \in \mathrm{PSh}(\mathcal{C})$ (with its natural maps $\mathcal{F}_i \to \mathcal{G}$) is their colimit in $\mathrm{PSh}(\mathcal{C})$, and $\mathcal{F} := \mathcal{G}^\sharp$ (with the natural maps $\mathcal{F}_i \to \mathcal{G} \to \mathcal{F}$) is their colimit in $\mathrm{Sh}(\mathcal{C})$. Because sheafification preserves finite limits, we see therefore that filtered colimits in $\mathfrak{X}$ preserve finite limits.

Clearly, each isomorphism is a monomorphism and an epimorphism. Thus, assume that $f\colon \mathcal{F} \to \mathcal{G}$ is a morphism in $\mathfrak{X} \cong \mathrm{Sh}(\mathcal{C})$, which is a monomorphism and an epimorphism. Given $Y \in \mathcal{C}$, the map $\mathcal{F}(Y) \to \mathcal{G}(Y)$ is therefore injective (because f is a monomorphism, and taking sections over Y preserves these). Take $s \in \mathcal{G}(Y)$. As f is an epimorphism, there exists a covering $\{Y_i \to Y\}_{i\in I}$ in $\mathcal{C}$ and sections $t_i \in \mathcal{F}(Y_i)$ with image $s_{|Y_i} \in \mathcal{G}(Y)$ under f. Given $i, j \in I$ with $Y_{ij} := Y_i \times_Y Y_j$, the sections $t_{i|Y_{ij}}, t_{j|Y_{ij}} \in \mathcal{F}(Y_{ij})$ agree. Indeed, because f is a monomorphism this can be checked after applying f, but

$$f(t_{i|Y_{ij}}) = s_{|Y_{ij}} = f(t_{j|Y_{ij}}).$$

Thus, the t_i glue to a section $t \in \mathcal{F}(Y)$, and necessarily $f(t) = s$ because $f(t)_{|Y_i} = s_{|Y_i}$ for each $i \in I$. This shows that $\mathcal{F}(Y) \to \mathcal{G}(Y)$ is bijective for any Y and hence f is an isomorphism.

Last statement on base change and effective epimorphisms are true in sets, but then they are also true in presheaves, and thus by exactness and left-adjointness of sheafification also for sheaves. □

Remark 3.12. Giraud's axioms characterize topoi among all categories: a category $\mathcal{C}$ is a topos if and only if it is generated under colimits by a set of objects, has universal colimits, disjoint coproducts and effective quotients ([AGV72, Éxposé IV, Théoème 1.2]). In fact, one defines that a collection $\{Y_i \to Y\}_{i\in I}$ of morphisms in a generating set of objects of $\mathcal{C}$ is a covering if and only if $\coprod_{i\in I} Y_i \to Y$ is an epimorphism. Then one checks that the category of sheaves on the resulting site is equivalent to $\mathcal{C}$.

Remark 3.13. From Remark 3.10 we can conclude that the functor

$$\Gamma(*, -)\colon \mathrm{CondSet}_\kappa \to \mathrm{Sets},\ X \mapsto X(*)$$

commutes with limits *and* all colimits. In more pitoresque terms, the set $X(*)$ is called the "underlying set" of the condensed set X (in Section 4.1 we will upgrade $X(*)$ to a topological space). From the perspective of topos theory, this example is quite special: usually coverings of the terminal object are much richer, e.g., for topological space.

We note another important case, where the sheafification of a certain colimit is not necessary.

Definition 3.14. (1) A site $(\mathcal{C}, \tau)$ is finitary if for each covering $\{Y_i \to Y\}_{i\in I}$ in τ the set I is finite.
(2) A topos $\mathfrak{X}$ is finitary if it is equivalent to $\mathrm{Sh}(\mathcal{C})$ for a finitary site.

Example 3.15. (1) We have seen that κ-condensed sets form a finitary topos.
(2) Let X be a spectral space, e.g., the spectrum of a ring. Then $\mathrm{Sh}(X)$ is finitary because one can realize it one the site $\mathcal{C} = \{U \subseteq X \text{ open and quasi-compact}\}$ (with *finite* open covers). Note that in this example, $\mathrm{Ouv}(X)$ is not necessarily a finitary site, yet $\mathrm{Sh}(X) \cong \mathrm{Sh}(\mathrm{Ouv}(X))$ is finitary.

Lemma 3.16. *Let $(\mathcal{C}, \tau)$ be a finitary site, and $\mathcal{F}_i \in \mathrm{Sh}(\mathcal{C})$, $i \in I$, a* filtered *system of sheaves with colimit $\mathcal{F}$. Then the natural map*

$$\varinjlim_{i\in I}(\mathcal{F}_i(X)) \to \mathcal{F}(X)$$

is bijective for any $X \in \mathcal{C}$.

Proof. Let $\mathcal{U} = \{X_j \to X\}_{j\in J}$ be a covering of some object $X \in \mathcal{C}$, and denote by $\varinjlim^p$ the colimit in *presheaves* on $\mathcal{C}$. We need to see that $\varinjlim\limits_{i\in I}^p \mathcal{F}_i$ is already a sheaf. As J is finite by assumption, the natural map

$$\varinjlim_{i\in I} \Gamma(X, \mathcal{F}_i) \cong \varinjlim_{i\in I} \Gamma(\mathcal{U}, \mathcal{F}_i) \to \Gamma(\mathcal{U}, \varinjlim_{i\in I}^p \mathcal{F}_i)$$

is bijective using that each $\mathcal{F}_i$ is a sheaf in the first step, and that filtered colimits commute with finite limits in the last step. □

Let us give a general construction for sheaves on a site.

Example 3.17. Let $\mathcal{C}$ be a site.

(1) Let $Y \in \mathcal{C}$. Then the representable presheaf

$$h_Y := \mathrm{Hom}_{\mathcal{C}}(-, Y) \in \mathrm{PSh}(\mathcal{C})$$

need not be a sheaf in general (if it is for any $Y \in \mathcal{C}$ the site is called subcanonical). By the Yoneda lemma we have a natural bijection

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_Y, \mathcal{F}) \cong \mathcal{F}(Y)$$

for any $\mathcal{F} \in \mathrm{PSh}(\mathcal{C})$. If $\mathcal{F}$ is a sheaf, we can conclude that

$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(h_Y^\sharp, \mathcal{F}) \cong \mathcal{F}(Y),$$

where the "representable sheaf" $h_Y^\sharp$ is the sheafification of h_Y. We call $h_Y^\sharp$ the free sheaf on Y. The functor $Y \to h_Y$ commutes with all limits (by definition of limits), and thus by exactness of sheafification $Y \to h_Y^\sharp$ commutes with *finite* limits.

(2) Note that if $\{Y_i \to Y\}_{i\in I}$ is a covering, then $\alpha\colon \coprod_{i\in I} h^*_{Y_i} \to h^*_Y$ is an epimorphism: given $Z \in \mathcal{C}$ and $s\colon h^*_Y(Z)$, then we can find a covering $\{Z_j \to Z\}_{j\in J}$ such that $s_{|Z_j}$ is represented by a morphism $s_j\colon Z_j \to Y$, i.e., $s_{|Z_j}$ is the image of s_j under $h_Y(Z_j) \to h_Y^\sharp(Z_j)$. Over the covering $\{Y_i \times_Y Z_j \to Z_j\}_{j\in J}$, $s_{j|Y_i\times_Y Z_j} \in h_Y(Y_i \times_Y Z_j)$ to a section $t_{ij} \in h_{Y_i}(Y_i \times_Y Z_j)$, namely the projection to Y_i. This shows that α is an epimorphism.

(3) Let $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(\mathcal{C})$. Then there exists a sheaf $\underline{\mathrm{Hom}}_{\mathrm{Sh}(\mathcal{C})}(\mathcal{F}, \mathcal{G})$ (the "internal Hom"), which is defined by the requirement that

$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(h_Y^\sharp, \underline{\mathrm{Hom}}_{\mathrm{Sh}(\mathcal{C})}(\mathcal{F}, \mathcal{G})) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(h_Y^\sharp \times \mathcal{F}, \mathcal{G})$$

for $Y \in \mathcal{C}$. Indeed, sending Y to the right hand formula defines a functor $\mathcal{C}^{\mathrm{op}} \to (\mathrm{Sets})$, and if $\{Y_i \to Y\}_{i\in I}$ is a covering, then (as we saw before) $\mathcal{M} := \coprod_{i\in I} h^\sharp_{Y_i} \to \mathcal{N} := h_Y^\sharp$ is an epimorphism. Hence, it is effective, and

$$\mathrm{coeq}(\mathcal{M} \times_{\mathcal{N}} \mathcal{M} \rightrightarrows \mathcal{M}) \cong \mathcal{N}.$$

By the universality of colimits (Theorem 3.11), this implies that $\beta\colon \mathcal{M} \times \mathcal{F} \to \mathcal{N} \times \mathcal{F}$ is also an effective epimorphism. Now,

$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(\mathcal{M} \times \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(\coprod_{i\in I} h^\sharp_{Y_i} \times \mathcal{F}, \mathcal{G}) \cong \prod_{i\in I} \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(h^\sharp_{Y_i} \times \mathcal{F}, \mathcal{G})$$

and similarly,

$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(\mathcal{M} \times_{\mathcal{N}} \mathcal{M} \times \mathcal{F}, \mathcal{G}) \cong \prod_{i,j\in I} \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(h^\sharp_{Y_i} \times_{h^\sharp_Y} h^\sharp_{Y_i} \times \mathcal{F}, \mathcal{G}),$$

so that the sheaf condition for $\underline{\mathrm{Hom}}_{\mathrm{Sh}(\mathcal{C})}(\mathcal{F}, \mathcal{G})$ unravels exactly to the statement that β is an effective epimorphism.

(4) We note that the natural isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(h_Y^\sharp, \underline{\mathrm{Hom}}_{\mathrm{Sh}(\mathcal{C})}(\mathcal{F}, \mathcal{G})) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(h_Y^\sharp \times \mathcal{F}, \mathcal{G})$$

generalizes to a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(\mathcal{H}, \underline{\mathrm{Hom}}_{\mathrm{Sh}(\mathcal{C})}(\mathcal{F}, \mathcal{G})) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(\mathcal{H} \times \mathcal{F}, \mathcal{G}).$$

Indeed, it suffices to write $\mathcal{H}$ as a colimit as in Lemma 3.18, and then to pull out the colimit in the first variable to a limit.

The category of sheaves is generated by the free sheaves.

Lemma 3.18. *Let $\mathcal{C}$ be a site and $\mathcal{F} \in \mathrm{Sh}(\mathcal{C})$. Then the map*

$$\mathcal{G} := \varinjlim_{Y \in \mathcal{C},\ s \in \mathcal{F}(Y) \cong \mathrm{Hom}(h_Y^\sharp, \mathcal{F})} h_Y^\sharp \to \mathcal{F}$$

is an isomorphism, which is natural in $\mathcal{F}$. In particular, $\mathrm{Sh}(\mathcal{C})$ *is generated by the sheaves $h_Y^\sharp$, $Y \in \mathcal{C}$, under colimits.*

Proof. The statement for $\mathrm{Sh}(\mathcal{C})$ follows from the one for $\mathrm{PSh}(\mathcal{C})$ by sheafification. Hence, assume that $\mathcal{C}$ is just a small category, and $\mathcal{F} \in \mathrm{PSh}(\mathcal{C})$. By the Yoneda lemma it suffices to show that for any $\mathcal{H} \in \mathrm{PSh}(\mathcal{C})$ the map

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\mathcal{F}, \mathcal{H}) \to \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\mathcal{G}, \mathcal{H}) \cong \varprojlim_{Y \in \mathcal{C},\ s \in \mathcal{F}(Y)} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_Y, \mathcal{H}) \cong \varprojlim_{Y \in \mathcal{C},\ s \in \mathcal{F}(Y)} \mathcal{H}(Y)$$

is a bijection. But starring at this last inverse limits reveals that it specifies exactly a natural transformation $\eta \colon \mathcal{F} \to \mathcal{H}$. □

We can know make the following important definitions.

Definition 3.19. Let $(\mathcal{C}, \tau)$ be a finitary site, and $\mathfrak{X} := \mathrm{Sh}(\mathcal{C})$. Let $f \colon Y \to X$ be a morphism in $\mathfrak{X}$.

(1) X is called quasi-compact if there exists a surjection $\coprod_{i=1}^n h_{X_i}^\sharp \to X$ for $X_1, \ldots, X_n \in \mathcal{C}$.
(2) f is called quasi-compact if for any $X' \to X$ with X' quasi-compact, the object $Y \times_X X'$ is quasi-compact.
(3) f is called quasi-separated if the diagonal $\Delta_{Y/X} \colon Y \to Y \times_X Y$ is quasi-compact.
(4) X is quasi-separated if for any quasi-compact object X' each morphism $X' \to X$ is quasi-compact, i.e., for any two quasi-compact objects X', X'' with morphisms to X, the fiber product $X' \times_X X''$ is quasi-compact.

We often adress quasi-compact to "qc" and quasi-separated to "qs".

Corollary 3.20. *Let $\mathfrak{X}$ be a finitary topos, and $X \in \mathfrak{X}$ qcqs. Then for any filtered colimit $\mathcal{F}_i \in \mathfrak{X}$, $i \in I$, with colimit $\mathcal{F}$ the natural map*

$$\varinjlim_{i \in I} \mathrm{Hom}_{\mathfrak{X}}(X, \mathcal{F}_i) \to \mathrm{Hom}_{\mathfrak{X}}(X, \mathcal{F})$$

is bijective.

Proof. Write $\mathfrak{X} = \mathrm{Sh}(\mathcal{C})$ for a finitary site $(\mathcal{C}, \tau)$. Let $Y := \coprod_{i \in I} h_{X_i}^\sharp \to X$ be a surjection with $X_1, \ldots, X_n \in \mathcal{C}$. Note that each $h_{X_i}^\sharp \times_X h_{X_j}^\sharp$ is quasi-compact because X is quasi-separated. In particular, we can find a surjection $Z \twoheadrightarrow Y \times_X Y$ with Z a finite disjoint union of representables. If $\mathcal{G} \in \mathfrak{X}$, we can conclude that $\mathrm{Hom}_{\mathfrak{X}}(X, \mathcal{G})$ is the equalizer of $\mathrm{Hom}_{\mathfrak{X}}(Y, \mathcal{G}) \rightrightarrows \mathrm{Hom}_{\mathfrak{X}}(Y \times_X Y, \mathcal{G})$, or equivalently of $\mathrm{Hom}_{\mathfrak{X}}(Y, \mathcal{G}) \rightrightarrows \mathrm{Hom}_{\mathfrak{X}}(Z, \mathcal{G})$. But then the claim follows from Lemma 3.16. □

3.2. **Interlude: compact Hausdorff spaces.** Before discussing profinite sets in detail, we provide a recollection of properties of compact Hausdorff spaces.

Lemma 3.21. *Let* Top *be the category of topological spaces with continuous maps, and let* CHaus $\subseteq$ Top *be the full subcategory of compact Hausdorff spaces.*

(1) CHaus $\subseteq$ Top *is stable under all limits.*
(2) *If* $f\colon Y \to X$ *is a quotient map of topological spaces with* Y *compact Hausdorff, then* X *is compact Hausdorff if and only if* $Y \times_X Y \subseteq Y \times Y$ *is closed.*
(3) *The functor* CHaus $\subseteq$ Top *admits a left-adjoint* $\beta\colon \mathrm{Top} \to \mathrm{CHaus}$*, called the Stone-Čech compactification* [Sta17, Tag 0908].

Proof. (i): We recall that it suffices to show that CHaus is stable under products and fiber products. That CHaus is stable under products is Tychonoff's theorem: any product of compact Hausdorff spaces is compact Hausdorff. We now recall that a topological space X is Hausdorff if and only if its diagonal $\Delta\colon X \to X \times X$ has closed image. From here, it follows easily that if $Y \to X,\ Z \to X$ are morphisms in CHaus, then their fiber product $Y \times_X Z \cong X \times_{X\times X} (Y \times Z)$ is closed in $Y \times Z$, and hence compact Hausdorff.

(ii): If X is compact Hausdorff, then $Y \times_X Y$ is closed in $Y \times Y$ as it is a base change of the diagonal $\Delta\colon X \to X \times X$. Conversely, assume that $Y \times_X Y \subseteq Y \times Y$ is closed. Let us note the following: if $K \subseteq Y$ is closed, then also $f^{-1}(f(K)) = \mathrm{pr}_2((K \times Y) \cap (Y \times_X Y))$ is closed, because the projection $\mathrm{pr}_2\colon Y \times Y \to Y$ is a closed map (using that Y is compact Hausdorff). In other words, the quotient map f is even a closed map. Let $x, x' \in X,\ x \neq x'$. We need to see that there exist disjoint open neighborhoods $U, U' \subseteq X$ of x, x'. Note that by the above observation the fibers $f^{-1}(f(y)),\ f^{-1}(f(y'))$ are compact subspaces of Y, and hence, there exist disjoint open neighborhoods V, V' of them. Now, $U := X \setminus (Y \setminus V)$, which is an open neighborhood of x, and $U' := X \setminus (Y \setminus V')$, which is an open neighborhood of x'. As $f^{-1}(U) \subseteq V,\ f^{-1}(U') \subseteq V'$, these neighborhoods are disjoint as desired.

(iii): Fix $X \in \mathrm{Top}$. If $f\colon X \to Y$ is a continuous map with $f(X) \subseteq Y$ dense and Y Hausdorff, then $|Y| \leq 2^{2^{|X|}}$: indeed, each point of Y is the unique limit of a filter[6] on X, and the set of filters on X is bounded by $|Y| \leq 2^{2^{|X|}}$. Hence, we can consider the *set* $f_i\colon X \to Y_i, i \in I$, of representatives for the isomorphism classes of Hausdorff topological spaces Y with a continuous map $f\colon X \to Y$ with dense image. We can now set $\beta(X)$ as the closure of the image of X in $\prod_{f_i\colon X\to Y_i} Y_i$. The universal property is easily checked. □

Example 3.22. Assume that X is a locally compact Hausdorff space. Then X is an open subset of its one-point-compactification $X \cup \{\infty\}$ (here, a system of neighborhoods of ∞ is given by $X \setminus K \cup \{\infty\}$ for $K \subseteq X$ compact). By the universal property of the Stone-Čech compactification there exists a continuous morphism $\varphi\colon \beta(X) \to X \cup \{\infty\}$. One checks that X is dense in $\varphi^{-1}(X)$ (as X is dense in $\beta(X)$), but also closed (as X is Hausdorff and $X \to \varphi^{-1}(X)$ is a section of the map $\varphi^{-1}(X) \to X$). Hence, $X \cong \varphi^{-1}(X)$ is open in $\beta(X)$. Note that this example is already quite wild if X is discrete, e.g., if $X = \mathbb{N}$, then $\beta(\mathbb{N}) \setminus \mathbb{N}$ has cardinality $2^{2^{|\mathbb{N}|}}$ [nLa]!

Inverse limits of non-empty compact Hausdorff spaces can be shown to be non-empty.

Lemma 3.23. *Let* $(K_i, f_{ij}\colon K_j \to K_i), i, j \in I, j \to i$*, be a cofiltered system of* non-empty *compact Hausdorff spaces. Then* $K := \varprojlim_{i\in I} K_i$ *is non-empty.*

Proof. We first note that if $L \in$ CHaus, and $Z_a \subseteq L,\ a \in A$, is a collection of closed subsets such that $\bigcap_{a\in B} Z_a \neq \emptyset$ for each finite subset $B \subseteq A$, then $\bigcap\limits_{a\in A} Z_a \neq \emptyset$. Indeed, $L = \bigcup_{a\in A} L \setminus Z_a$ is a union of open subspaces, and by compactness of L a finite subcovering exists.

Now recall that K is the subspace

$$\{(k_i)_{i\in I} \in \prod_{i\in I} K_i \mid f_{ij}(k_k) = k_i \text{ for all } j \to i \text{ in } I\}$$

of the topological product $\prod_{i\in I} K_i$. For any finite subset $J \subseteq I$ we consider the subspace

$$K_J := \{(k_i)_{i\in I} \in \prod_{i\in I} K_i \mid f_{ij}(k_j) = k_i \text{ for all } j \to i \text{ in } J\}.$$

Because each K_i is Hausdorff, and the f_{ij} are continuous each subspace K_J is closed (this reduces to the observation that the graph of f_{ij} is closed in $K_j \times K_i$, which itself follows from closedness of the diagonal of K_i). By definition, $K = \bigcup_{J\subseteq I \text{ finite}} K_J$, and hence it suffices to show that each

[6] A filter on a set S is a subset $\mathcal{B}$ of the power set of S, such that $S \in \mathcal{B}$, $\emptyset \notin \mathcal{B}$, $\mathcal{B}$ is stable under finite intersections and if $T \in \mathcal{B}$ and $T \subseteq T'$, then $T' \in \mathcal{B}$.

K_J is non-empty (using that $K_J \subseteq K_{J'}$ if $J' \subseteq J$). Using that I is cofiltered and finiteness of J, there exists some $i_0 \in I$ such that i_0 admits a unique morphism to any $j \in J$. Then one can pick any $k \in K_{i_0}$ and set $k_j := f_{ji_0}(k)$ for $j \in J$, and any element in $\prod\limits_{i \in I \setminus J \cap \{i_0\}} K_i \neq \emptyset$, to see that $K_J \neq \emptyset$. □

Corollary 3.24. *Let $K_i, i \in I$ be a cofiltered system of compact Hausdorff, and assume that for $j \to i$ the morphism $f_{ij}\colon K_j \to K_i$ is surjective.*

1. *For each $i \in I$ the projection $f_i\colon K := \varprojlim_{j \in I} K_j \to K_i$ is surjective.*
2. *A morphism $L \to K$ from a compact Hausdorff space L is surjective if and only if for each $i \in I$ the composition $L \to K \to K_i$ is surjective.*

Proof. For (i): Let $k \in K_i$, and for $j \to i$ let $Z_j := f_{ij}^{-1}(k)$, which is a closed subspace of K_j. Note that $Z_j \neq \emptyset$ as f_{ij} is surjective. Then $f_i^{-1}(k) = \varprojlim\limits_{j \to i} Z_j \neq \emptyset$ by Lemma 3.23.

For (ii): If $g\colon L \to K$ is surjective, then $L \to K \to K_i$ surjective by (i). Assume that each $g_i\colon L \to K_i$ is surjective. Then $g^{-1}(k) = \bigcap_{i \in I} g_i^{-1}(f_i(k))$ is a cofiltered limit of non-empty compact Hausdorff spaces, and hence non-empty by Lemma 3.23. □

Remark 3.25. If $M_i, i \in I$, is a *countable* cofiltered system of non-empty sets with surjective transition functions $f_{ij}\colon M_j \to M_i$, then $\varprojlim_{i \in I} M_i \neq \emptyset$ (using induction). This fails if the f_{ij} are not surjective (e.g., $M_i = \mathbb{Z} \setminus \{0\}$, and f_{ij} multiplication by 2), or if I is uncountable (e.g., let I be the poset of finite subsets of $\mathbb{R}$, and $M_i := \{\text{injective maps } i \to \mathbb{Z}\}$, then $\varprojlim_{i \in I} M_i$ identifies with the set of injections $\mathbb{R} \to \mathbb{Z}$, and hence is empty).

Lemma 3.26 ([Sta17, Tag 08ZN])**.** *Let K be a compact Hausdorff space, and $x \in K$. Then the connected component of x is the intersection of all open and closed subsets Z_a, $a \in A$, containing x.*

Proof. It suffices to show that $S := \bigcap_{a \in A} Z_a$ is connected (as it contains the connected component of x because any Z_a is open and closed in K). Assume that $S = B \coprod C$ with B, C open and closed in S. Then B, C are closed in K. As K is compact Hausdorff, there exists open subsets $U, V \subseteq K$ with $B \subseteq U, C \subseteq V$ and $U \cup V = \emptyset$. Now, $K' := K \setminus U \cup V$ is closed in K, and hence compact. As $S \cap K' == \bigcap_{a \in A} Z_a \cap K' = \emptyset$, it follows from Lemma 3.23, that there exists some $a \in A$ with $Z_a \cap K' = \emptyset$, i.e., $Z_a \subseteq U \cap V$. Now, $Z_a = (U \cap Z_a) \coprod (V \cap Z_a)$ is a decomposition into open and closed subsets, and x must lie in them, say $x \in U \cap Z_a$. But this implies that $S \subseteq U \cap Z_a$ cannot intersect $V \cap Z_a$, and hence $C = \emptyset$. □

Quasi-compact spaces admit the following characterization.

Lemma 3.27 (Kuratowski's theorem)**.** *Let K be a topological space. The following are equivalent:*

1. *K is quasi-compact*
2. *for any topological space X the map $f\colon X \times K \to X$, $(x, k) \mapsto x$ is closed.*

Proof. Assume that K is quasi-compact, and let $Z \subseteq X \times K$ be closed. Let $x \in X \setminus f(Z)$. This implies that $\{x\} \times K \subseteq X \times K \setminus Z$. As K is quasi-compact, we can find an open subset $V \subseteq X$ with $\{x\} \times K \subseteq V \times X \subseteq X \times K \setminus Z$ (this follows from the definition of the product topology, and is sometimes called the tube lemma, [Sta17, Tag 005N]). But this implies that $V \subseteq X \setminus f(Z)$ is an open neighborhood of x.

For the converse, we refer to [Sta17, Tag 005P]. □

We can note the following corollary.

Lemma 3.28 (Whitehead's theorem)**.** *Let $f\colon Y \to X$ be a quotient map of topological spaces, and K locally quasi-compact, i.e., each point has a fundamental system of quasi-compact neighborhoods. Then*

$$g\colon Y \times K \to X \times K,\ (y, k) \mapsto (f(y), k)$$

is a quotient map.

Proof. Let $V \subseteq X \times K$ be a subset such that $g^{-1}(V) \subseteq Y \times K$ is open. Let $(x, k) \in V$. We need to find some open subsets $U \subset X, W \subseteq K$ such that $U \times W \subseteq V$ and $(x, k) \in U \times W$. Let $y \in Y$ with $f(y) = x$. As $g^{-1}(V)$ is open, we can find some open subset $W \subseteq K$ such that $k \in W$ and $\{y\} \times W \subseteq g^{-1}(V)$. Shrinking W we may assume that $\{y\} \times \overline{W} \subseteq g^{-1}(V)$. Here, $\overline{W}$ denotes a quasi-compact neighborhood of k. We note that if $z \in Y$ with $\{z\} \times \overline{W} \subseteq g^{-1}(V)$, then $f^{-1}(f(z)) \times \overline{W} \subseteq g^{-1}(V)$. Indeed, $g(f^{-1}(f(z)) \times \overline{W}) = g(\{z\} \times \overline{W})$. set $U := \{x' \in$

$X \mid f^{-1}(x') \times \overline{W} \subseteq g^{-1}(V)\}$. It suffices to see that U is open in X (as $x \in U$). As $f\colon Y \to X$ is a quotient map, it suffices to check that $f^{-1}(U) = \{\, y' \in Y \mid f^{-1}(f(y')) \times \overline{W} \subseteq g^{-1}(V)\} = \{y' \in Y \mid \{y'\} \times \overline{W} \subseteq g^{-1}(V)\}$. Let $\pi\colon Y \times \overline{W} \to Y$ be the projection. Then π is closed by Lemma 3.27, and $f^{-1}(U) = Y \setminus (\pi(Y \times \overline{W} \setminus g^{-1}(V)))$ is open in Y. □

3.3. **Profinite sets.** Profinite sets are very special compact Hausdorff spaces. We recall their definition.

Definition 3.29. A profinite set S is a compact Hausdorff space, which is totally disconnected, i.e., the only connected subspaces of S are singletons.

Actually, profinite sets arise "in nature".

Example 3.30.
(1) $S = \{0\} \cup \{1/n \mid n \in \mathbb{N}\} \subseteq [0,1]$ is a profinite set for the subspace topology. Note that $S \cong \mathbb{N} \cup \{\infty\}$ is homeomorphic to the one-point-compactification of $\mathbb{N}$.
(2) Let $X_1 := [0,1]$ be the closed unit interval. We can consider the inverse system
$$\ldots \to X_n := [0, 1/2^n] \cup [1/2^n n, 2/2^n] \cup \ldots \cup [2^n - 1/2^n, 1] \to X_{n-1} \to \ldots \to X_1 = [0,1]$$
of compact Hausdorff spaces. Then $S := \varprojlim_{n\in\mathbb{N}} X_n$ is a profinite set: by Lemma 3.21 it is compact Hausdorff, and it is totally disconnected. Indeed, assume that $Z \subseteq X$ is closed and connected. Then the image of Z in X_n lies in $[a_n/2^n, a_n + 1/2^n]$ for some $a_n \in \{0, 2^n - 1\}$. Because $1/2^n \to 0$ for $n \to \infty$, Z cannot contain two distinct points as those have to map to different points in some X_n.
(3) If S is a profinite set, and $Z \subseteq S$ is closed, then Z is profinite. Indeed, Z is compact Hausdorff, and each non-empty connected subset of Z is also connected in S, and hence a singleton.

Lemma 3.31. *Let S be a topological space. Then S is a profinite set if and only if $S = \varprojlim_{i\in I} S_i$ is a cofiltered limit (in topological spaces) of finite, discrete sets S_i. In particular, each point $s \in S$ has a basis of neighborhoods which are open* and *closed.*

Proof. Assume that $S = \varprojlim_{i\in I} S_i$ with S_i finite, discrete. Then Lemma 3.21 implies that S is compact Hausdorff. If $U \subseteq S$ is connected, and non-empty, then the image of U in each S_i is a point. This implies that U is a singleton as $S = \varprojlim_{i\in I} S_i$. Conversely, assume that S is a profinite set. We consider the cofiltered partially ordered set I of quotient maps $S \to S_i$ with S_i finite and discrete. There exists a natural continuous morphism $\varphi\colon S \to \varprojlim_{i\in I} S_i$ of compact Hausdorff spaces. By Lemma 2.1 it suffices to show that φ is bijective. By Corollary 3.24 it is surjective because each $S \to S_i$ is. Note that for $x \in S$ the inverse image $\varphi^{-1}(\varphi(x))$ is exactly the intersection of all open and closed subsets containing x. By Lemma 3.26, therefore $\varphi^{-1}(\varphi)(x) = \{x\}$ as S is totally disconnected and φ is injective. For the final assertion note that for each $s \in S = \varprojlim_{i\in I} S_i$ with projections $f_i\colon S \to S_i$, and S_i finite, discrete, the inverse images $f_i^{-1}(f_i(s))$ form a system of open and closed neighborhoods (by the definition of the limit topology over a cofiltered diagram). This finishes the proof. ☐

We now give equivalent description of the category of profinite sets. Note that the two other categories in Proposition 3.32 don't refer in any way to topological spaces.

Proposition 3.32 (Stone duality)**.** *The following categories are equivalent:*
(1) *The full subcategory* Prof $\subseteq$ CHaus *of profinite sets.*
(2) *The pro-category* Pro(Fin) *of finite sets, i.e., the category with objects* $"\varprojlim_{i\in I}" S_i$ *given by cofiltered diagrams $S_i, i \in I$, of finite sets, and morphisms*
$$\mathrm{Hom}_{\mathrm{Pro(Fin)}}("\varprojlim_{i\in I}" S_i, "\varprojlim_{j\in J}" T_j) := \varprojlim_{j\in J} \varinjlim_{i\in I} \mathrm{Hom}_{\mathrm{Fin}}(S_i, T_j).$$
(3) *The opposite of the category of Boolean algebras R, i.e., commutative rings with $x^2 = x$ for all $x \in R$.*

Proof. Let $S = \varprojlim_{i\in I} S_i$, $T = \varprojlim_{j\in J} T_j$ two cofiltered limits of finite discrete sets. Then
$$\mathrm{Hom}_{\mathrm{Prof}}(S, T) = \varprojlim_{j\in J} \mathrm{Hom}_{\mathrm{Prof}}(S, T_j) \cong \varprojlim_{j\in J} \varinjlim_{i\in I} \mathrm{Hom}_{\mathrm{Prof}}(S_i, T_j),$$
which implies by Lemma 3.31 that Prof is equivalent to Pro(Fin).

Given a profinite set S we set $A := \mathrm{Cont}(S, \mathbb{F}_2)$ as the ring of $\mathbb{F}_2$-valued continuous functions on S. Then A is clearly a Boolean algebra. Given conversely a Boolean algebra R, we define $S := \mathrm{Spec}(R)$ as the spectrum of A. Using basic results in scheme theory, one can now argue that S is a spectral space, which does not admit any specializations (because R is Boolean), and hence is a profinite set. Avoiding this recursion into spectral spaces, one can also argue as follows: let R be a Boolean algebra. The following statements are easily checked:

- R is an $\mathbb{F}_2$-algebra as $2x = (2x)^2 = 4x$,

- each finite $\mathbb{F}_2$-subvector space $V \subseteq R$ generates a *finite* $\mathbb{F}_2$-subalgebra (because $\mathbb{F}_2[x_1, \ldots, x_n]/(x_1^2 - x_1, \ldots, x_n^2 - x_n)$ is finite),
- if R is finite, then $R \cong C(T, \mathbb{F}_2) \cong \mathbb{F}_2^T$ for some finite set T (indeed, R is a finite dimensional, reduced $\mathbb{F}_2$-algebra, and hence a finite product of fields, which necessarily have to be isomorphic to $\mathbb{F}_2$ as they are Boolean),
- the category of finite Boolean algebras is anti-equivalent to the category of finite sets via the functors $R \mapsto \mathrm{Hom}_{\mathrm{Rings}}(R, \mathbb{F}_2)$ and $T \mapsto C(T, \mathbb{F}_2)$.

Thus, R is the filtered colimit of its finite Boolean subalgebras, and these finite Boolean algebras are $C(T, \mathbb{F}_2)$ for some finite set T. This implies that $\mathrm{Hom}_{\mathrm{Rings}}(R, \mathbb{F}_2)$ is naturally a profinite set. One checks that the functors $S \mapsto C(S, \mathbb{F}_2)$ and $R \mapsto \mathrm{Hom}_{\mathrm{Rings}}(R, \mathbb{F}_2)$ are mutually inverse. □

We used the following lemmata.

Lemma 3.33. *Let $S := \varprojlim_{i \in I} S_i$ be a cofiltered limit of finite discrete sets. Let M be a set, viewed as a discrete topological space. Then the natural map*

$$\varinjlim_{i \in I} \mathrm{Hom}_{\mathrm{cts}}(S_i, M) \to \mathrm{Hom}_{\mathrm{cts}}(S, M)$$

is bijective.

Proof. Injectivity is easy. So let us assume that $\varphi \colon S \to M$ is a continuous map. Then φ is locally constant, but then φ must be constant on the fibers of $S \to S_i$ for some i, and thus descents to a map $\psi \colon S_i \to M$, which is then automatically continuous. □

Remark 3.34. We leave it as an exercise to check that a morphism $T \to S$ of profinite sets is injective (surjective) if and only if $\mathrm{Cont}(S, \mathbb{F}_2) \to \mathrm{Cont}(T, \mathbb{F}_2)$ is surjective (injective).

We can now prove Theorem 2.5.

Theorem 3.35. (1) *A topological space S is a profinite set if and only if $S \cong \varprojlim_{i \in I} S_i$ is a cofiltered limit of finite sets S_i.*
(2) *Each compact Hausdorff space admits a surjection from a profinite set.*
(3) *The full subcategory* Prof $\subseteq$ Top $:= \{$*topological spaces*$\}$ *of profinite sets is stable under all limits.*

Proof. (i) is Lemma 3.31.

(ii): Let $K \in$ CHaus. We set I as the cofiltered poset of finite open coverings $\{U_{ij} \to K\}_{i \in J_i}$, $i \in I$. For $i \in I$ set $K_i := \coprod_{j \in J_i} \overline{U_{ij}}$, where $\overline{U_{ij}}$ is the closure of U_{ij} in K. Then each K_i is compact Hausdorff, and also $S := \varprojlim_{i \in I} K_i$. By Corollary 3.24 the morphism $S \to K$ is surjective. It suffices to see that S is totally disconnected. But this is clear: given two different points $s, t \in S$, their images x, y in K_i must be different for some $i \in I$. Passing to a refinement $i' \to i$ of the covering given by i, we can arrange that s, t map to different $\overline{U_{i',j}}$ of $K_{i'}$. Indeed, we can refine i in such a way that x, y lie in a single $U_{i'j}$ of the covering i'.

(iii): This follows from stability of profinite sets under products and Example 3.30 stability of Prof under closed subsets. □

Remark 3.36. Note that in the proof of Theorem 3.35 we can assume that the U_{ij} lie in some fixed basis for the topology of K. In particular, if K is second-countable, then I will be countable. Note furthermore that the argument in the proof of Theorem 3.35 also shows that

$$S \cong \varprojlim_{i \in I} K_i \cong \varprojlim_{i \in I} J_i,$$

via the map sending $\overline{U_{ij}} \subseteq K_i$ to $j \in J_i$. In particular, each second-countable compact Hausdorff space admits a surjection from a profinite set, which is a *countable* inverse limits of finite sets, that is, an ω_1-profinite set in the terminology of Definition 3.51.

Corollary 3.37. *Let S be a set, viewed as a discrete topological spaces. Then*

$$\beta S \cong \varprojlim_{S \to S_i \text{ finite quotient}} S_i$$

is a profinite set. Moreover, each surjection $S' \to \beta S$ with $S' \in$ CHaus *admits a continuous splitting $\beta S \to S'$.*

Proof. We first show the last assertion: we can pick a set-theoretic section $S \to S'$, and then by the universal property of the Stone-Čech-compactification this map extends to a continuous morphism $\beta S \to S'$, which has to be a section of $S' \to \beta S$, because this is true on the dense subset $S \subseteq \beta S$. Now, we check that βS is profinite. Indeed, this follows by picking any surjection $S' \to \beta S$ with

S' profinite (such a surjection exists by Theorem 3.35), and then any section $\beta S \to S'$ will realize βS as a closed subset of S'. This implies that βS is profinite. Knowing that βS is profinite implies that $\beta S \varprojlim_{i\in I} S_i$ with S_i finite, discrete quotients of βS. Note that by density of S in βS, $S \to S_i$ is still surjective. Moreover, any finite quotient of S extends to a finite, discrete quotient of βS by the universal property of βS. This finishes the proof. □

Remark 3.38. Corollary 3.37 shows that βS for a set S is a *projective object* in CHaus, also called an extremally disconnected or Gleason space. Moreover, for any $K \in \mathrm{CHaus}$ the map $\beta(K^{\mathrm{disc}}) \to K$ shows that each CHaus admits a (canonical) surjection from a projective objects. Note, however, that $|\beta(K^{\mathrm{disc}})| = 2^{2^{|K|}}$ is very big. For completeness we note that $C(\beta S, \mathbb{F}_2) = \mathrm{Hom}(S, \mathbb{F}_2) = \mathbb{F}_2^S$ is the Boolean algebra associated with βS.

Lemma 3.39. *Let $S \in \mathrm{CHaus}$. Then S is a profinite set if and only if for each open cover $U_i, i \in I$, of S the map $\coprod_{i\in I} U_i \to S$ admits a continuous splittings.*

Proof. Let $s, t \in S$ with $s \neq t$. As in the proof of Theorem 3.35 we can find an open cover $U_i,\ i \in I$, of S with s, t contained in the same U_i. But then the existence of a continuous splitting of $\coprod_{i\in I} U_i \to S$ implies that s, t must lie in different connected components. In other words, S is necessarily profinite. Assume now that S is profinite, and $U_i \to S,\ i \in I$. Because each point Lemma 3.31 has a neighborhood basis consisting of open and closed subsets, we may assume (by refining) that I is finite and the U_i are open and closed. But then $V_i := U_i \setminus \cup_{i\neq j} U_j$ define a covering $V_i,\ i \in I$, of S with the V_i pairwise disjoint. But then $\coprod_{i\in I} V_i \to S$ is an isomorphism, and thus admits trivially a splitting. □

We can also note the following description of closed subspaces of profinite sets.

Lemma 3.40. *Let $S \in \mathrm{Prof}$ and $Z \subseteq S$ closed. Then Z is the intersection of all its compact and open neighborhoods.*

Proof. Write $S = \varprojlim_{i\in I} S_i$, and let $Z_i \subseteq S_i$ be the image of Z. Then $Z \cong \varprojlim_{i\in I} Z_i$ because Z is closed in S (indeed by Corollary 3.24 Z is dense in $\varprojlim_{i\in I} Z_i \subseteq S$). But, then the inverse images of Z_i under the projections $S \to S_i$ define a cofinal system of compact and open neighborhoods of Z, with intersection Z. □

3.4. **Interlude: ordinals and cardinals.** As explained in Section 2 we have to bound the size of profinite sets in order to use topos theory in condensed mathematics. In this subsection we gather some definitions and results on ordinals and cardinals.

Definition 3.41. A set α is an ordinal if every element of α is also a subset of α, and if α is well-ordered for the relation: $x \leq y$ for α if $x \subseteq y$.

For example,

$$0 := \emptyset,\ 1 := \{0\} = \{\emptyset\},\ 2 := \{0,1\} = \{\emptyset, \{\emptyset\}\},\ \ldots$$

Ordinals are very rigid. We leave the following assertions as exercises.

Remark 3.42.
(1) Each element of an ordinal is an ordinal, and if α, β are ordinals, then $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.
(2) Each well-ordered set is uniquely order-isomorphic to a unique ordinal, and by the axiom of choice each set is in bijection with an ordinal. In particular, ordinals form a class and not a set.
(3) Via "$\subseteq$" the class of ordinals is well-ordered. In fact, the lower bound of a set of ordinals is their intersection
(4) Given an ordinal α, its successor is the ordinal $\alpha + 1 := \alpha \cup \{\alpha\}$, and a successor ordinal α is one of the form $\beta + 1$ for some ordinal β.
(5) If α is not a successor ordinal, it is called a limit ordinal, and $\alpha = \bigcup_{\beta\in\alpha} \beta$. The first limit ordinal is $\alpha = \{0, 1, 2, \ldots\}$.
(6) The first uncountable ordinal ω_1 is the union of all countable ordinals, and ω_1 is a limit ordinal.
(7) Because ordinals are well-ordered one can use transfinite induction to prove statements on ordinals: Let P be a property of ordinals and suppose $P(\alpha)$ is true, whenever $P(\beta)$ is true for any $\beta < \alpha$. Then $P(\alpha)$ is true for any α. Similarly, one can use transfinite recursion to construct functions on ordinals.

Ordinals provide a convenient way to define cardinals.

Definition 3.43. Given a set A its cardinality is the least ordinal α such that there exists a bijection $A \cong \alpha$. A cardinal κ is an ordinal, which is the cardinality of some set A.

Remark 3.44.
(1) $\aleph_0 := \omega$ is a cardinal as is $\aleph_1 := \omega_1$. More generally, for a cardinal κ let κ^+ be its successor cardinal, i.e., the least cardinal $> \kappa$ (this exists because the class of ordinals is well-ordered). Then

$$\aleph_1 = \aleph_0^+,\ \aleph_2 := \aleph_1^+,\ \ldots.$$

(2) One can define $\aleph_\alpha$ for any ordinal α using transfinite induction. If $\alpha = 0$, then $\aleph_0 := \omega$. If $\alpha = \beta + 1$, then $\aleph_\alpha = \aleph_\beta^+$, and $\aleph_\alpha := \bigcup_{\beta<\alpha} \aleph_\beta$ if α is a limit ordinal.
(3) Given cardinals κ, λ, then $\kappa + \lambda$ is the cardinality of $\kappa \coprod \lambda$, $\kappa \cdot \lambda$ is the cardinality of $\kappa \times \lambda$, and κ^λ is the cardinality of $\mathrm{Hom}(\lambda, \kappa)$.
(4) If κ, λ are infinite, then $\kappa + \lambda = \kappa \times \lambda = \max\{\kappa, \lambda\}$ (the maximum is taken with respect to the ordering of ordinals).
(5) For any cardinal κ, one has $\kappa^+ \leq 2^\kappa$.

Definition 3.45. A strong limit cardinal κ is a cardinal, such that $2^\lambda < \kappa$ for any cardinal $\lambda < \kappa$.

Lemma 3.46. *There exist arbitrary large strong limit cardinals.*

Proof. The proof uses Beth numbers, which are defined using transfinite recursion:

- $\beth_0 := \aleph_0$,
- $\beth_{\alpha+1} := 2^{\beth_\alpha}$,
- $\beth_\alpha := \sup\{\beth_\beta \mid \beta < \alpha\}$ if α is a limit ordinal.

By transfinite induction, one can show that $\beth_\beta < \beth_\alpha$ if $\beth < \alpha$, and that $\alpha < \beth_\alpha$. Now, if α is a limit ordinal, then $\kappa := \beth_\alpha$ is a strong limit cardinal. Indeed, if $\lambda < \kappa$, then there exists some $\beta < \alpha$ with $\kappa < \beth_\beta$ (by the definition of the supremum). But then $2^\kappa < 2^{\beth_\beta} = \beth_{\beta+1} < \kappa$ (because $\beta + 1 < \alpha$). □

Remark 3.47. The generalized continuum hypothesis is the statement that $\beth_\alpha = \aleph_\alpha$ for any ordinal α.

Finally, we can relate ordinals to profinite sets.

Definition 3.48. An ordinal space is a topological space given by an ordinal α with its order topology, i.e., the open sets of α are unions of sets of the form $[0, x) := \{\beta \in \alpha \mid \beta < x\}$, $(x, y) := \{\beta \in \alpha \mid x < \beta < y\}$, $(x, \infty) := \{\beta \in \alpha \mid x < \beta\}$ for $x, y \in \alpha$.

Note the similarity with the topology on the real numbers $\mathbb{R}$. In particular, we can also consider the closed intervals $[x, y]$ for $x, y \in \alpha$. If $\alpha = \omega + 1$, then for example $[0, \omega]$ is the one-point-compactification $\mathbb{N} \cup \{\infty\}$ of $\mathbb{N}$.

Lemma 3.49. *An ordinal space α is Hausdorff and locally profinite, i.e., admits an open cover by profinite sets. It is a profinite set if and only if α is a successor ordinal or zero.*

Proof. Assume that $x, y \in \alpha$, $x \neq y$. Wlog $x < y$. Then (x, ∞) and $[0, x + 1)$ are disjoint open subsets separating x and y. Thus α is Hausdorff. Assume that $x \in \alpha$. Then $[0, x] \subseteq \alpha$ is closed as its complement is (x, α), but it is also open because $[0, x] = [0, x + 1)$. By transfinite induction it is easy to see that $[0, x]$ is compact: any neighborhood of x needs to contain some $(y, x]$ for some $y < x$ (if $x \neq 0$), and by induction $[0, y]$ is compact. The last assertion is easy. □

Remark 3.50. It can be shown that $\beta\omega_1 = \omega_1 + 1$ and that any continuous function $\omega_1 \to \mathbb{R}$ is eventually constant.

3.5. **κ-condensed sets.** Fix an uncountable cardinal κ. We can now turn to the discussion of κ-condensed sets (following [CS23, Lecture 2] closely).

Let $S = \varprojlim_{i\in I} S_i$ be a profinite set with $S_i,\ i \in I$, finite and discrete.

Definition 3.51. (1) The *size* of S is $\kappa(S) := |S|$.
(2) The *weight* of S is $\lambda(S) := |\mathrm{Cont}(S, \mathbb{F}_2)|$.
(3) Assume that κ is an uncountable cardinal S is called a κ-profinite set, or κ-light, if $\lambda < \kappa$.
(4) If $\kappa = \omega_1$, then we abbreviate κ-light to light, i.e., S is light if $\lambda \leq \omega$.

If S is clear from the context, we abbreviate $\kappa(S), \lambda(S)$ to κ, λ.

Note that $\mathrm{Cont}(S, \mathbb{F}_2)$ is in bijection with open and closed subsets $U \subseteq S$. Thus, the definition in Definition 3.51 recovers Definition 2.6.

For example, $\mathbb{N} \cup \{\infty\}$ and $\prod_{\mathbb{N}}\{0, 1\}$ are light profinite set as is $\mathbb{Z}_p \cong \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n$ for a prime p, while $\prod\limits_{I}\{0, 1\}$ is not light if I is uncountable.

Remark 3.52. Write $S = \varprojlim_{i\in I} S_i$. If $\lambda < \infty$, then S is finite and we can assume that $I = \{*\}$, and thus potentially $|I| < \lambda = 2^\kappa$. If however, λ is infinite, then the smallest possible $|I|$ is λ. Indeed, we may assume that $S \to S_i$ is surjective for any $i \in I$, and that $S_j \to S_i$ is an isomorphism if and only if $i = j$. Then $|I| \leq \lambda$ follows easily from the definition. Conversely, $\lambda \leq |I|$ because each open and closed subset of S is pulled back from some S_i (here we use that λ is infinite, which implies that $|I| = \sum_{i\in I} 2^{|S_i|}$).

We mention some examples.

Example 3.53. (1) If S is a finite discrete set, then $\kappa = |S|$ while $\lambda = 2^\kappa$.
(2) If $S = \mathbb{N} \cup \{\infty\} \cong \varprojlim_{n\in\mathbb{N}}\{0, 1, \ldots, n, \infty\}$, then $\kappa = \lambda = \omega$.
(3) If $S = \prod_{\mathbb{N}}\{0, 1\}$ is the Cantor set, then $\kappa = 2^\omega$ and $\lambda = \omega$.
(4) If $S = \beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$, then $\mathrm{Cont}(S, \mathbb{F}_2) \cong \mathrm{Cont}(\mathbb{N}, \mathbb{F}_2) \cong \{\text{subsets of } \mathbb{N}\}$ and thus $\lambda = 2^\omega$, while $\kappa = 2^{2^\omega}$.

In particular, uncountable profinite sets such as the Cantor set can be light. Size and weight are related.

Lemma 3.54. *We have $\lambda \leq 2^\kappa$ and $\kappa \leq 2^\lambda$. Moreover, if κ is infinite, then $\lambda \leq \kappa$.*

Proof. We have
$$\lambda = |A| \leq |\mathrm{Hom}_{\mathrm{Sets}}(S, \mathbb{F}_2)| = 2^\kappa$$
for $A := \mathrm{Cont}(S, \mathbb{F}_2)$ and
$$\kappa = |\mathrm{Hom}_{\mathrm{Rings}}(A, \mathbb{F}_2)| \leq |\mathrm{Hom}_{\mathrm{Sets}}(A, \mathbb{F}_2)| = 2^\lambda.$$
Assume that κ is infinite (or equivalently that λ is infinite). Let I be the set of finite subsets of S. Note that $|I| = |S|$ because κ is infinite. For any finite subset $J \subseteq S$ we can consider a finite discrete quotient $S \to S_J$ of S such that $J \to S_J$ is injective. We can moreover arrange that the quotients S_J form a cofiltered system. Then $S \to \varprojlim_{J\in I} S_J$ is injective (by construction) and surjective (by Corollary 3.24). By Remark 3.52, we see that $\lambda \leq |I| = \kappa$. □

Corollary 3.55. *Assume that κ' is a strong limit cardinal. Then S is κ'-light if and only if $\kappa(S) < \kappa'$.*

Proof. This follows from the definition of a strong limit cardinal and Lemma 3.54: if $\lambda < \kappa'$, then $\kappa \leq 2^\lambda < \kappa'$, and if $\kappa < \kappa'$, then $\lambda \leq 2^\kappa < \kappa'$. □

Lemma 3.56. *We denote by* $\mathrm{Prof}_\kappa \subseteq \mathrm{Prof}$ *the full subcategory of κ-profinite sets. Then* $\mathrm{Prof}_\kappa \subseteq \mathrm{Prof}$ *is stable under countable limits, and passage to closed subsets/quotients.*

Proof. By Proposition 3.32 the Boolean algebra associated with a limit of profinite sets, is the colimit of the associated Boolean algebras. Now colimits of Boolean algebras are calculated in the category of rings (in fact quotient/subalgebras/tensor products of Boolean algebras are again Boolean because being Boolean means that the Frobenius $x \mapsto x^2$ on an $\mathbb{F}_2$-algebra is the identity, which is stable under these operations), and countable colimits of Boolean algebras of size $< \kappa$ will again be of type κ. The last statement follows from Remark 3.34. □

Let us now recall the definition of a κ-condensed set.

Definition 3.57. (1) We define a Grothendieck topology on Prof_κ by saying that a family $\{S_i \to S\}_{i\in I}$ is a covering if I is finite and $\coprod_{i\in I} S_i \to S$ is surjective.
(2) A κ-condensed set is a sheaf on Prof_κ.

(3) We let $\mathrm{CondSet}_\kappa := \mathrm{Sh}(\mathrm{Prof}_\kappa)$ be the topos of κ-condensed sets.

From Theorem 3.11 we can conclude that $\mathrm{CondSet}_\kappa$ has all limits and colimits, and that filtered colimits commute with finite limits.

For a strong limit cardinal κ one can note an easier description of CondAb_κ.

Definition 3.58. Let κ be a strong limit cardinal. We let $\mathrm{ExtDisc}_\kappa \subseteq \mathrm{Prof}_\kappa$ be the full sub category of projective object, i.e., the subcategory of retracts of the κ-profinite sets $\beta(T)$, where T is a discrete set of cardinality $< \kappa$.

Lemma 3.59. *Let κ be a strong limit cardinal. Then the restriction along $\mathrm{ExtDisc}_\kappa \to \mathrm{Prof}_\kappa$ defines an equivalence*

$$\Phi\colon \mathrm{CondSet}_\kappa \to \mathrm{Fun}^\times(\mathrm{ExtDisc}_\kappa^{\mathrm{op}}, \mathrm{Sets}),$$

where the right hand side denotes functors $\mathrm{ExtDisc}_\kappa^{\mathrm{op}} \to \mathrm{Sets}$ preserving finite products.

Proof. Clearly Φ is well-defined. We can construct a functor in the reverse direction: given $\mathcal{F} \in \mathrm{Fun}^\times(\mathrm{ExtDisc}_\kappa^{\mathrm{op}}, \mathrm{Sets})$, we define

$$\Psi(\mathcal{F}) \in \mathrm{CondSet}_\kappa$$

to be the functor $S \in \mathrm{Prof}_\kappa \mapsto \varprojlim_{T \to S,\ T \in \mathrm{ExtDisc}_\kappa} \mathcal{F}(T)$. One checks that $\Psi(\mathcal{F})$ is indeed a sheaf. Namely, given a surjection $S' \to S$, and a morphism $T \to S$ with $T \in \mathrm{ExtDisc}_\kappa$, there exists a lift $T \to S'$ of $T \to S$, and from here one checks that $\mathrm{eq}(\Psi(\mathcal{F})(S') \rightrightarrows \Psi(\mathcal{F})(S' \times_S S')) \cong \Psi(\mathcal{F})(S)$. Moreover, $\Psi \circ \Phi$, $\Phi \circ \Psi$ are naturally isomorphic to the identity functors. When checking that $\Psi \circ \Phi \cong \mathrm{Id}_{\mathrm{CondAb}_\kappa}$ one needs to use that each $S \in \mathrm{Prof}_\kappa$ admits a surjection $T \to S$ with $T \in \mathrm{ExtDisc}_\kappa$. This is implied by κ being a strong limit cardinal as we may take $T = \beta(S^{\mathrm{disc}})$. □

Remark 3.60. The critical property needed for Lemma 3.59 is the fact that the condition on surjections in Prof_κ in the definition of a condensed set becomes obsolete when restricting to $\mathrm{ExtDisc}_\kappa$. Indeed, thanks to Lemma 3.5 the sheaf condition is vacuous for coverings that admit a splitting.

Via sites Lemma 3.59 can be reinterpreted as follows: we can equip $\mathrm{ExtDisc}_\kappa$ with the structure of a site where a collection $\{T_i \to T\}_{i \in I}$ of morphisms is a covering if and only I is finite and the map $\coprod_{i \in I} T_i \to T$ is an isomorphism. Sheaves for this Grothendieck topology are exactly functors preserving finite products. Now, Lemma 3.59 is an instance of the observation that sheaves on a site can

4. Topological spaces as condensed sets

In this section we want to explain in detail how to pass between topological spaces and condensed sets through the functor

$$\underline{(-)}\colon \mathrm{Top} \to \mathrm{CondSet}_\kappa,\ X \mapsto \mathrm{Hom}_{\mathrm{cts}}(S, X)$$

from Lemma 2.8. In full generality, this matter is quite subtle, e.g., the functor is not invariant under enlarging κ. The strongest results hold for *metrizable* topological spaces and light condensed sets, and we will discuss this in more detail in Section 4.4.

4.1. **Topological spaces and κ-condensed sets.** Let κ be an uncountable cardinal. We will denote by $* \in \mathrm{Prof}_\kappa$ the profinite set with a single point. We note that by definition $\underline{S} = \mathrm{Hom}_{\mathrm{Prof}_\kappa}(-, S)$ for a κ-profinite set Prof_κ, and thus by the Yoneda lemma we have the following consequence: given $T \in \mathrm{CondSet}_\kappa$, then each $\varphi \in T(S)$ defines a morphism $\underline{S} \to T$ of condensed sets, and thus a map $S = \underline{S}(*) \to T(*)$.

It is very common to identify an object in a category $\mathcal{C}$ with its associated Hom-functor, and thus to view the Yoneda lemma as the identity functor. To avoid confusion, we will only do this in later subsections, and for now we write $\underline{S} \in \mathrm{CondSet}_\kappa$ instead of just S, for $S \in \mathrm{Prof}_\kappa$.

Lemma 4.1. *The functor $\underline{(-)}\colon \mathrm{Top} \to \mathrm{CondSet}_\kappa,\ X \mapsto \underline{X}$ has a left adjoint $T \mapsto T(*)_{\mathrm{top}}$, which can be described as follows: the underlying set of $T(*)_{\mathrm{top}}$ is $T(*) = \mathrm{Hom}_{\mathrm{CondSet}}(*, T)$, and the topology on $T(*)_{\mathrm{top}}$ is the quotient topology for the morphism $\coprod_{\varphi \in T(S)} S \to T(*)$.*

Proof. We have to see that there is a functorial bijection

$$\mathrm{Hom}_{\mathrm{CondSet}_\kappa}(T, \underline{X}) \cong \mathrm{Hom}_{\mathrm{Top}}(T(*)_{\mathrm{top}}, X)$$

for $T \in \mathrm{CondSet}_\kappa$ and $X \in \mathrm{Top}$. Note that for $S \in \mathrm{Prof}_\kappa$, the map $\underline{X}(S) \to \prod_{s \in S} \underline{X}(\{s\}) = \prod_{s \in S} X$ is injective (it identifies with the inclusion of continuous functions $S \to X$ to all functions). This implies that a morphism $T \to \underline{X}$ of κ-condensed sets, i.e., a compatible system of maps $T(S) \to \underline{X}(S)$ for $S \in \mathrm{Prof}_\kappa$, is equivalently given by a map $T(*) \to X$, such that for each element in $T(S)$, the induced morphism $S \to T(*) \to X$ is continuous. But these are exactly the continuous morphisms $T(*)_{\mathrm{top}} \to X$. □

Fully faithfulness of $(-)$ translates in the question for which $X \in \mathrm{Top}$ the morphism $\underline{X}(*)_{\mathrm{top}} \to X$. This leads to the following class of topological spaces, that we will study in more detail in Section 4.2.

Definition 4.2. Let κ be an uncountable cardinal, and let X be a topological space.

(1) $X^{\kappa-\mathrm{cg}}$ is the topological space with underlying set X equipped with the quotient topology for the morphism $\coprod_{S \to X \text{ cont.},\ S \in \mathrm{Prof}_\kappa} S \to X$.[7]
(2) X is called κ-compactly generated if the natural morphism $X^{\kappa-\mathrm{cg}} \to X$ is an isomorphism.

We can check fully faithfulness of $(-)$ on κ-compactly generated topological spaces.

Lemma 4.3. *The functor $(-)\colon \mathrm{Top}^{\kappa-\mathrm{cg}} \to \mathrm{CondSet}_\kappa$ is fully faithful, and its left adjoint $Y \mapsto Y(*)_{\mathrm{top}}$ commutes with finite products.*

Proof. We need to see that the morphism $\underline{X}(*)_{\mathrm{top}} \to X$ is a homeomorphism if X is κ-compactly generated. But this follows from the definition. To check that $Y \mapsto Y(*)_{\mathrm{top}}$ commutes with products, take $Y, Z \in \mathrm{CondSet}_\kappa$ and surjections $Y' \to Y,\ Z' \to Z$ of disjoint unions Y', Z' of κ-profinite set. Then $(Y' \times Z')(*)_{\mathrm{top}} \to (Y \times Z)(*)_{\mathrm{top}}$ is a quotient map (by the definition of the topology), and moreover for each $S \in \mathrm{Prof}_\kappa$ with a continuous map to $Y(*)_{\mathrm{top}} \times Z(*)_{\mathrm{top}}$ there exists a covering $S' \to S$ and a lift to a morphism $S' \to (Y' \times Z')(*)_{\mathrm{top}} = Y'(*)_{\mathrm{top}} \times Z'(*)_{\mathrm{top}}$ (here we use that the product topology for κ-profinite sets is κ-compactly generated). But this implies that the set $(Y \times Z)(*)_{\mathrm{top}} = Y(*)_{\mathrm{top}} \times Z(*)_{\mathrm{top}}$ with its quotient topology from $(Y' \times Z')(*)_{\mathrm{top}}$ is the product in $\mathrm{Top}^{\kappa-\mathrm{cg}}$, as desired (Remark 4.16). □

Thus on the large category of κ-compactly generated space (Lemma 4.27) the functor from topological spaces to $\mathrm{CondSet}_\kappa$ is fully faithful.

Remark 4.4. Let $\mathfrak{X}$ be a topos. Then there exists a unique (up to unique isomorphism) morphism $\mathfrak{X} \to \mathrm{Sets},\ Y \mapsto \mathrm{Hom}_{\mathfrak{X}}(*, Y)$, where $*$ is a terminal object of $\mathfrak{X}$. Indeed, the left adjoint necessarily needs to send a point set to a terminal object of $\mathfrak{X}$.[8] We can describe this adjunction differently for $\mathfrak{X} = \mathrm{CondSet}_\kappa$. Let $(-)^{\mathrm{disc}}\colon \mathrm{Sets} \to \mathrm{Top}$ be the functor equipping a set with the discrete topology.

[7] If κ is a strong limit cardinal, then this definition agrees with from [Sch, Paragraph before Remark 1.6].

[8] This crucially uses that the left adjoint of a morphism of topoi preserves finite limits.

The composition $\mathrm{Sets} \xrightarrow{(-)^{\mathrm{disc}}} \mathrm{Top} \xrightarrow{\underline{(-)}} \mathrm{CondSet}_\kappa$ is then the left adjoint to $\mathrm{CondSet}_\kappa \to \mathrm{Sets}$, $T \mapsto T(*)$ (sending a condensed set to its "underlying set"). Note that the functor $\mathrm{Sets} \to \mathrm{CondSet}_\kappa$ is fully faithful.

The functor $\underline{(-)}$ also preserves some categorical constructions.

Lemma 4.5. *The functor* $\underline{(-)}\colon \mathrm{Top} \to \mathrm{CondSet}_\kappa$ *preserves*

(1) *limits,*
(2) *coproducts, i.e., disjoint unions.*

Proof. The preservation of limits follows directly from the definition, and the preservation of coproducts is a direct consequence of the description of the coproducts of condensed sets: given $\mathcal{F}_i \in \mathrm{CondSet}_\kappa$, $i \in I$ their coproduct $\coprod_{i\in I} \mathcal{F}_i$ is the functor

$$S \in \mathrm{Prof}_\kappa \mapsto \{(S = \coprod_{i\in I} S_i,\ s_i \in \mathcal{F}_i(S_i) \mid S_i \subseteq S \text{ open and closed}\}. \qquad \square$$

In general, the functor $\underline{(-)}$ does not commute with a filtered colimit, even along injections Remark 4.23. We mention some positive results.

Lemma 4.6. *Let X be a topological space, which is the filtered union $X = \bigcup_{i\in I} X_i$ of subspaces $X_i \subseteq X$. The natural morphism $\bigcup_{i\in I} \underline{X_i} \to \underline{X}$ is an isomorphism of κ-condensed sets if one of the following conditions is satisfied:*

(1) *the interiors of X_i cover X,*
(2) *I is* countable *and X is Hausdorff.*
(3) *X is a CW-complex, and the X_i are subcomplexes.*

Proof. We need to see that for any $S \in \mathrm{Prof}_\kappa$, the natural map

$$\varinjlim_{i\in I} \mathrm{Hom}_{\mathrm{cts}}(S, X_i) \to \mathrm{Hom}_{\mathrm{cts}}(S, X)$$

is bijective. This is clear in the first case because S is quasi-compact, so its image is contained in some interior of some X_i.

Thus, assume the second case. We assume $I = \mathbb{N}$, and that there exists a continuous map $f\colon S \to X$, such that $f(S)$ is not contained in X_n for any $n \in \mathbb{N}$. Then we obtain a sequence of points $s_i \in S$ such that $f(s_i) \in X_i \setminus X_{i-1}$ (potentially reindexing and omitting some of the X_i's). As S is compact Hausdorff, there exists a map $\beta(\mathbb{N}) \to S$, which extends the map $i \mapsto s_i$. We may then replace S by $\beta(\mathbb{N})$. Set $T := \{f(s_i) \mid s_i \in \mathbb{N}\}$. Assume that $x \in X \setminus T$. Then $x \in X_j$ for some j. For $i > j$, the intersection $T \cap X_i$ is finite (by construction of the s_i). As X_i is Hausdorff, we find an open neighborhood $U_i \subseteq X_i$ of x such that $U_i \cap T = \emptyset$. As X_i has the subspace topology of X_{i+1}, we can find $U_{i+1} \subseteq X_{i+1}$ an open neighborhood of x with $U_{i+1} \cap X_i = U_i$, and then we may also assume that $U_{i+1} \cap T = \emptyset$ by removing the finitely many point $T \cap X_{i+1}$. These properties imply that $U := \bigcup_{i\in\mathbb{N}} U_i$ is open in X (by definition of the colimit topology), and hence an open neighborhood of x that misses T. Hence, $f^{-1}(U) = \mathbb{N} = \emptyset$, and hence $f^{-1}(U) = \emptyset$ as $\mathbb{N} \subseteq \beta(\mathbb{N})$ is dense. This shows that $f(S) = T$. Now, $T \subseteq X$ is discrete because for any point $x \in T \cap X_j$, we can as before inductively find open neighborhoods V_i of $x \in X_i$ with $X_i \cap U_{i+1} = U_i$ and $T \cap X_i = \{x\}$ for $i > j$. But then T is discrete and compact (being contained in the compact image of S in X). This is a contradiction.

Now, assume that X is a CW-complex, and the X_i are subcomplexes. By the closure finiteness property of CW complexes, each morphism $S \to X$ for $S \in \mathrm{Prof}_\kappa$ factors through a finite subcomplex. This reduces the statement to (2). $\square$

We can also compare internal Hom's of (κ-compactly generated) topological spaces with that of κ-condensed sets, Definition 4.14, Example 3.17.

Lemma 4.7. *Assume that X, Y are topological spaces, with X κ-compactly generated. Then*

$$\underline{\underline{\mathrm{Hom}}_{\mathrm{Top}}(X, Y)} \cong \underline{\mathrm{Hom}}_{\mathrm{CondSet}_\kappa}(\underline{X}, \underline{Y})$$

Proof. Let $S \in \mathrm{Prof}_\kappa$. Note that if X is κ-compactly generated, then X is compactly generated, and $X \times S \cong (X \times S)^{\mathrm{cg}}$ by Lemma 4.12. Then

$$\begin{aligned}
\underline{\mathrm{Hom}}_{\mathrm{CondSet}_\kappa}(\underline{X}, \underline{Y})(S) &= \mathrm{Hom}_{\mathrm{CondSet}_\kappa}(\underline{S} \times \underline{X}, \underline{Y}) \\
&\underset{\textit{Lemma 4.1}, \textit{Lemma 4.3}}{=} \mathrm{Hom}_{\mathrm{Top}}(S \times \underline{X}_{\mathrm{top}}(*), \underline{Y}) \\
&\underset{X\ \kappa\text{-comp. gen.}}{=} \mathrm{Hom}_{\mathrm{Top}}(S \times X, Y) \\
&\underset{\textit{Lemma 4.15}}{=} \mathrm{Hom}_{\mathrm{Top}}(S, \underline{\mathrm{Hom}}_{\mathrm{Top}}(X, Y)) \\
&= \underline{\underline{\mathrm{Hom}}_{\mathrm{Top}}(X, Y)}(S),
\end{aligned}$$

and this isomorphism is natural in S. Hence, the claim follows. □

4.2. **Interlude: κ-compactly generated topological spaces.** Let κ be an uncountable cardinal. In this subsection, we want to collect some material on κ-compactly generated topological spaces (following the account [Str] for compactly generated topological spaces).

We recall their definition from Definition 4.2:

Definition 4.8. Let X be a topological space.

(1) $X^{\kappa-\mathrm{cg}}$ is the topological space with underlying set X equipped with the quotient topology for the morphism $\coprod_{S\to X \text{ cont.},\ S\in \mathrm{Prof}_\kappa} S \to X$.
(2) X is called κ-compactly generated if the natural morphism $X^{\kappa-\mathrm{cg}} \to X$ is an isomorphism.
(3) X is called κ-weakly Hausdorff, if for any continuous map $f\colon S\to X$ with $S\in \mathrm{Prof}_\kappa$, the image $f(S)\subseteq X$ is closed.

We note the following equivalent characterizations.

Lemma 4.9. *Let X be a topological space. The following are equivalent:*

(1) *X is κ-compactly generated,*
(2) *there exists a collection $S_i\in \mathrm{Prof}_\kappa,\ i\in I$, and a quotient map $f\colon \coprod_{i\in I} S_i\to X$.*

Proof. The forward implication is the definition. Assume there exists such a quotient map f. Then f factors visibly over $X^{\kappa-\mathrm{cg}}$, and thus $X^{\kappa-\mathrm{cg}}\to X$ is a quotient map, and thus an isomorphism. □

The functor $(-)^{\kappa-\mathrm{cg}}$ is a right adjoint.

Lemma 4.10. *Let $\mathrm{Top}^{\kappa-\mathrm{cg}}\subseteq \mathrm{Top}$ be the full subcategory of κ-compactly generated spaces.*

(1) *The fully faithful inclusion $\mathrm{Top}^{\kappa-\mathrm{cg}}\to \mathrm{Top}$ has the right adjoint $(-)^{\kappa-\mathrm{cg}}$.*
(2) *$\mathrm{Top}^{\kappa-\mathrm{cg}}$ has all colimits and all limits. Colimits are calculated in Top, while limits are calculated as follows: take the limit in Top, and then apply $(-)^{\kappa-\mathrm{cg}}$.*
(3) *$\mathrm{Top}^{\kappa-\mathrm{cg}}$ is stable under quotients.*

Proof. Let $f\colon X\to Y$ be a continuous map of topological spaces, and assume that X is κ-compactly generated. We need to see that f promotes to a continuous map $X\to Y^{\kappa-\mathrm{cg}}$. By definition, we may reduce to the case that $X=S\in \mathrm{Prof}_\kappa$. But then the map $S\to Y$ lifts to $\coprod_{S'\to Y} S'\to Y$, with S' running through all κ-profinite sets with a continuous morphism to Y. By the definition of $Y^{\kappa-\mathrm{cg}}$ we can conclude that $S\to Y^{\kappa-\mathrm{cg}}$ is continuous as desired.

The second statement is a formal consequence. The stability under quotients follows from the stability under coequalizers, and the observation that a coequalizer $\mathrm{coeq}(Y\rightrightarrows X)$ in Top with $X\in \mathrm{Top}^{\kappa-\mathrm{cg}}$ does not change if Y is replaced by $Y^{\kappa-\mathrm{cg}}$. □

Remark 4.11. We spell out how to access finite products in $\mathrm{Top}^{\kappa-\mathrm{cg}}$: let $X,Y\in \mathrm{Top}^{\kappa-\mathrm{cg}}$, and choose the quotient maps $\coprod_{i\in I} S_i\to X$, $\coprod_{j\in J} S_j\to Y$ with $S_i,S_j\in \mathrm{Prof}_\kappa$ and I,J running through all continuous maps to X resp. Y. Then the quotient topology for

$$\coprod_{(i,j)\in I\times J} S_i\times S_j\to X\times Y$$

is the topology on $(X\times Y)^{\kappa-\mathrm{cg}}$. Indeed, each morphism $S\to X\times Y$ with $S\in \mathrm{Prof}_\kappa$, factors as $S\to S\times S\to X\times Y$, which shows that the quotient topology agrees with κ-compactly generated one (one can also use Remark 4.16).

There are cases when the topology on the product need not be changed.

Lemma 4.12. *Let $X\in \mathrm{Top}^{\kappa-\mathrm{cg}}$ and let $K\in \mathrm{CHaus}$. Assume that K admits a surjection from a κ-profinite set. Then the natural morphism*

$$(X\times K)^{\kappa-\mathrm{cg}}\to X\times K$$

is a homeomorphism.

The same assertion holds true if K is replaced by a κ-locally compact space, i.e., i.e., a locally compact space, which admits an open covering by the interiors of compact Hausdorff spaces as in Lemma 4.12.

Proof. Pick a surjection $S\to K,\ S\in \mathrm{Prof}_\kappa$, and a quotient map $\coprod_{i\in I} S_i\to X,\ S_i\in \mathrm{Prof}_\kappa$. By Lemma 3.28 the map

$$\coprod_{i\in I}(S_i\times K)\to X\times K$$

is a quotient map. As the same is true for each $S_i\times S\to S_i\times K$. We can conclude that the topological product $X\times K$ is already κ-compactly generated, and hence agrees with $(X\times K)^{\kappa-\mathrm{cg}}$. □

Remark 4.13. The product $X \times Y$ of first countable topological spaces X, Y is again first countable. But any first countable topological space is κ-compactly generated (as it admits a quotient map from a disjoint union of $\mathbb{N} \cup \{infty\}$'s), and so $(X \times Y)^{\kappa-\mathrm{cg}} \cong X \times Y$ in this case as well. Note that this argument generalizes to countable products of first-countable spaces.

The κ-compactly generated topology is useful in discussion internal Hom's. We first clarify what we mean by this.

Definition 4.14. Let X, Y be topological spaces. We set

$$\underline{\mathrm{Hom}}_{\mathrm{Top}}(X, Y) := \{f : X \to Y \mid f \text{ continuous}\},$$

and equip it with the compact-open topology, which is the topology generated by subsets $U(u, K, V) := \{f \in \underline{\mathrm{Hom}}_{\mathrm{Top}}(X, Y) \mid f \circ u(K) \subseteq V\}$ for K a compact Hausdorff space, $u : K \subseteq X$ continuous, and $V \subseteq Y$ open.[9]

If X is Hausdorff, the map $u : K \to X$ in Definition 4.14 may assumed to be injective as this does not change the compact-open topology.

We note that $\underline{\mathrm{Hom}}_{\mathrm{Top}}$ is naturally a functor

$$\mathrm{Top} \times \mathrm{Top} \to \mathrm{Top},\ (X, Y) \mapsto \underline{\mathrm{Hom}}_{\mathrm{Top}}(X, Y),$$

i.e., composing with a continuous map $Y \to Y'$ yields a continuous map $\underline{\mathrm{Hom}}_{\mathrm{Top}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathrm{Top}}(X, Y')$, and similarly for precomposition with a continuous map $X' \to X$.

We can note the following general lemma.

Lemma 4.15. *Let X, Y, Z be topological spaces. Assume that X, Y are compactly generated. Then the map*

$$\mathrm{Hom}_{\mathrm{Top}}((X \times Y)^{\mathrm{cg}}, Z) \to \mathrm{Hom}_{\mathrm{Top}}(X, \underline{\mathrm{Hom}}_{\mathrm{Top}}(Y, Z)),\ f \mapsto (x \mapsto (y \mapsto f(x, y)))$$

is a well-defined bijection.

Here, $(X \times Y)^{\mathrm{cg}}$ denotes $X \times Y$ with the compactly generated topology (defined as the κ-compactly generated topology without the restriction imposed through κ).

Proof. Assume that $f : (X \times Y)^{\mathrm{cg}} \to Z$ is a continuous map. Let $g : X \to \underline{\mathrm{Hom}}_{\mathrm{Top}}(Y, Z),\ x \mapsto (y \mapsto f(x, y))$. As Y is compactly generated, $f(x, -)$ is continuous for each $x \in X$. Let $u : K \to Y$ be a continuous map from a compact Hausdorff space, and $V \subseteq Z$ open. Let $U(u, K, Y) \subseteq \underline{\mathrm{Hom}}_{\mathrm{Top}}(Y, Z)$ be the open set from Definition 4.14. We need to see that $g^{-1}(U(u, K, Y)) = \{x \in X \mid f(x, u(K)) \subseteq V\} \subseteq X$ is open. Note that $h : X \times K \to (X \times Y)^{\mathrm{cg}}$ is continuous (using Lemma 4.12 for some large enough κ). Now, $W := h^{-1} \circ f^{-1}(V) \subseteq X \times K$ is open in $X \times K$, and as the projection $p : X \times K \to X$ is closed (Lemma 3.27), $X \setminus p(X \times K \setminus W)$ is open in X. But this open subset is exactly $g^{-1}(U(u, K, Y))$.

For the converse, assume that $g : X \to \underline{\mathrm{Hom}}_{\mathrm{Top}}(Y, Z)$ is continuous. We need to see that $f : X \times Y \to Z,\ (x, y) \mapsto g(x)(y)$ is continuous. Let $K \in \mathrm{CHaus}$ with a continuous map $K \to X \times Y$. We need to see that the composition $K \to X \times Y \xrightarrow{f} Z$ is continuous. Factoring the first map as $K \xrightarrow{\Delta} K \times K \to X \times Y$, we see that we can replace $X = Y$ by K, and assume that X, Y are compact Hausdorff. Let $V \subseteq Z$ be open and $(x, y) \in f^{-1}(V)$. As $g(x)$ is continuous, there exists a compact neighborhood K of y in Y with $g(x)(K) \subseteq V$. We can now consider the open subset $U := U(K \to Y, K, V) \subseteq \underline{\mathrm{Hom}}_{\mathrm{Top}}(Y, Z)$. Then $W := g^{-1}(U) \subseteq X$ is open and $(x, y) \in W \times K \subseteq f^{-1}(V)$. Indeed, if $x \in W$ by construction of K, and if $(w, y) \in W \times Y$, then $f(w, y) = g(w)(y) \in V$ as $g(w) \in U$. Thus, f is continuous as desired. □

Remark 4.16. Let $Y, X', X \in \mathrm{Top}^{\mathrm{cg}}$ and $X' \to X$ a quotient map, then $f : (X' \times Y)^{\mathrm{cg}} \to (X \times Y)^{\mathrm{cg}}$ is again a quotient map. Indeed, it suffices to show that f is an effective epimorphism. But this can be tested by mapping to $Z \in \mathrm{Top}^{\mathrm{cg}}$, and then $\mathrm{Hom}_{\mathrm{Top}}((X \times Y)^{\mathrm{cg}}, Z) \cong \mathrm{Hom}_{\mathrm{Top}}(X, \underline{\mathrm{Hom}}_{\mathrm{Top}}(Y, Z))$ by Lemma 4.15. This implies the claim.

[9]If $X \subseteq \mathrm{C}$ is open, and $Y = \mathrm{C}$, then the compact-open topology generalizes the topology of uniform convergence that is often used in complex analysis (for example).

4.3. **Quasi-compact and quasi-separated condensed sets.** Recall that in Definition 3.19 we defined quasi-compactness and quasi-separatedness in a finitary topos. In this subsection, we study them for condensed sets.

As a first consequence, we can show that the functor $\underline{(-)}$ preserves subobjects of κ-compactly generated spaces. More precisely, one has the following.

Lemma 4.17. *Let X be a κ-compactly generated space, and let $Z \subseteq X$ be a closed subspace.*

(1) *Z is κ-compactly generated,*
(2) *$\underline{Z} \to \underline{X}$ is a quasi-compact injection of κ-condensed sets,*
(3) *each* quasi-compact *injection $Y \to X$ is isomorphic to the inclusion $\underline{W} \to \underline{X}$ for a closed subspace $W \subseteq X$*

Proof. (i): This follows from the fact that, because Z is closed in X, the base change $Z \times_X (-)$ preserves topological quotients and κ-profinite sets.

(ii): We need to see that for any quasi-compact object $T \in \mathrm{CondSet}_\kappa$ the fiber product $T \times_{\underline{X}} \underline{Z}$ is again quasi-compact. Note that the base change $(-) \times_{\underline{X}} \underline{Z}$ preserves epimorphisms (in any topos base change preserves epimorphisms). By the definition of quasi-compactness we may therefore assume that $T = S$ is a profinite set (identified with the condensed set $\underline{S} = h_S$ through the Yoneda embedding). But then $S \times_{\underline{X}} \underline{Z} \cong \underline{S \times_X Z}$ because $\underline{(-)}$ commutes with limits, and $S \times_X Z$ is a profinite set (being closed in S). In particular, $S \times_{\underline{X}} \underline{Z}$ is a quasi-compact condensed set as desired.

(iii): Assume that $Y \to \underline{X}$ is a quasi-compact injection. Assume first that $X = S$ is a κ-profinite set, and pick a κ-profinite set S' with a surjection $S' \to Y$ (this is possible because S and that the morphism $Y \to S$ are quasi-compact, so Y is quasi-compact). We get a factorization $S' \to Y \to := \underline{W} \to S$ of condensed sets, where $W := Y(*)_{\mathrm{top}}$. Note that as $S' \to Y$ is surjective, we have $W = \mathrm{Im}(S' \to S)$ (as sets). But this implies (by Lemma 2.1) that W must already carry the subspace topology of S, and that it is closed in S. In particular, W is a κ-profinite set. Moreover, the morphism $S = \underline{S'} \to \underline{W}$ is an epimorphism of κ-condensed sets (as follows from the definition of the Grothendieck topology on Prof_κ [10]). This implies that $Y \to W$ is an epimorphism. But as $Y \to S$ is a monomorphism, $Y \to W$ must be an isomorphism (using Theorem 3.11). Now, assume that X is a general κ-compactly generated space. We have the surjection $f\colon \coprod_{i\in I} S_i \to X$ of topological spaces, where I runs through all continuous morphisms $S_i \to X$ from κ-profinite sets. With this choice of I it is clear that $\coprod_{i\in I} \underline{S_i} \to \underline{X}$ is an epimorphism of κ-condensed sets. This implies that $g\colon \coprod_{i\in I} Y \times_{\underline{X}} S_i \to Y$ is an epimorphism. By the previous case, each $Y \times_{\underline{X}} S_i$ is the inclusion of a closed subset $W_i \subseteq S_i$. Moreover, we can note that $\underline{W_i \times_X S_j} \cong Y \times_X (S_i \times_X S_j) \cong \underline{S_i \times_X W_j}$, and thus $W_i \times_{S_i} (S_i \times_X S_j) = W_j \times_{S_j} (S_i \times_X S_j)$. This implies, that there exists a unique closed subspace $W \subseteq X$ such that $W \times_X S_i = W_i$ (because f is a topological quotient morphism). From the surjectivity of g, we can then conclude that $\underline{W} = Y$, and that necessarily $W = Y(*)_{\mathrm{top}}$. ☐

Motivated by Lemma 4.17 one can define that a morphism $Y \to X$ of κ-condensed sets is an *open immersion* if for each $S \to X$ with $S \in \mathrm{Prof}_\kappa$ pullback $Y \times_X S \to S$ is of the form $\underline{U} \to S = \underline{S}$ for an open subset $U \subseteq S$.

We can now relate qcqs condensed sets to classical objects.

Lemma 4.18. *The functor $\underline{(-)}$ induces an equivalence between:*

(1) *the category* CHaus_κ *of compact Hausdorff spaces K, which admit a surjection from a κ-profinite set,*[11]
(2) *qcqs objects in* $\mathrm{CondSet}_\kappa$.

Proof. We first check that if $S \to K$ is a surjection from a κ-profinite set to a compact Hausdorff space, then the morphism $S = \underline{S} \to \underline{K}$ is an epimorphism. It suffices to see that $S' \times_{\underline{K}} S \to S'$ is an epimorphism for any map $S' \to \underline{K}$ from some $S' \in \mathrm{Prof}_\kappa$. Note that $S' \times_{\underline{K}} S \cong \underline{S' \times_K S}$, and that $S' \times_K S \subseteq S' \times S$ is closed, hence profinite. But as $S \to K$ is surjective, $S' \times_K S \to S'$ is surjective, and hence a covering in the Grothendieck topology in Prof_κ. But any covering defines an epimorphism of (associated representable) sheaves.

We can conclude that $S \to \underline{K}$ is even an effective epimorphism (Theorem 3.11), i.e., $\underline{K}$ is the quotient of S by the equivalence relation $R := S \times_{\underline{K}} S \subseteq S \times S$. In particular, it follows that $\underline{K}$

[10] One can observe the following general statement: Let $(\mathcal{C}, \tau)$ be a site, and $\{Y_i \to Y\}_{i\in I}$ a collection of morphisms in $\mathcal{C}$. Then $\coprod_{i\in I} h^\sharp_{Y_i} \to h^\sharp_Y$ is an epimorphism in the topos $\mathfrak{X} = \mathrm{Sh}(\mathcal{C})$ if and only if the collection $\{Y_i \to Y\}_{i\in I}$ can be refined by a covering $\{Z_j \to Y\}_{j\in J}$ in $\mathcal{C}$, Example 3.17.

[11] E.g., if κ is a strong limit cardinal, then $K \in \mathrm{CHaus}$ lies in CHaus_κ if and only if $|K| < \kappa$ (this follows because $\beta(K^{\mathrm{disc}})$ has cardinality $< \kappa$).

is qcqs because it is the quotient of the qcqs condensed set S along a *quasi-compact* equivalence relation $R \subseteq S \times S$.

Assume conversely that $X \in \mathrm{CondSet}_\kappa$ is qcqs, and pick an epimorphism $S \to X$ with $S \in \mathrm{Prof}_\kappa$ (this is possible as X is qc). Now, the equivalence relation $R := S \times_X S \subseteq S \times S$ is quasi-compact, because X is qs. This implies (using that $S \times S$ is quasi-separated) that the morphism $R \to S \times S$ is quasi-compact. By Lemma 4.17, this implies that $R = \underline{W} \subseteq S \times S$, for a closed subspace $S \times S$, which necessarily must be a profinite set. Note that $W \subseteq S \times S$ is again an equivalence relation. By Lemma 3.21 we can conclude that the quotient $K := S/W$ is a compact Hausdorff space (which trivially admits a surjection from a κ-profinite set), and from the previous discussion we know that

$$\underline{K} \cong \mathrm{coeq}(\underline{W} \rightrightarrows \underline{S}) \cong \mathrm{coeq}(R \rightrightarrows \underline{S}) \cong X$$

through the natural maps. More precisely, we use effectivity of epimorphisms in $\mathrm{CondSet}_\kappa$ in the last isomorphism (Theorem 3.11) □

We now want to study quasi-separated condensed sets. Let us note some stability properties of quasi-separated objects.

Lemma 4.19. *Let $\mathfrak{X}$ be a finitary topos.*

(1) *Let $Y \to X$ be a monomorphism, with X quasi-separated. Then Y is quasi-separated.*
(2) *Filtered colimits of quasi-separated objects along* injective *transition maps are quasi-separated.*
(3) *Each quasi-separated object X is the filtered colimit of its qcqs subobjects.*

Proof. Write $\mathfrak{X} = \mathrm{Sh}(\mathcal{C})$ for a finitary site $\mathcal{C}$. If $Y \to X$ is a monomorphism, then $S \times_Y T = S \times_X T$ for any $S, T \in \mathfrak{X}$ over Y. From this formula, first two statements follow from Corollary 3.20.

Assume now that X is quasi-separated. There exists a surjection $\coprod_{i \in I} Y_i \to X$ with $Y_i \in \mathcal{C}$ (in particular, each Y_i is quasi-compact). Thus, it suffices to see that for $Y \in \mathfrak{X}$ quasi-compact, the image Z of a morphism $Y \to X$ is a qcqs subobject of X. Being a subobject of X, Z is quasi-separated. Being a quotient of Y, Z is also quasi-compact. This finishes the proof. □

Hence, one can describe quasi-separated objects via certain filtered colimits. Recall that if $\mathcal{C}$ is a category, then its Ind-completion $\mathrm{Ind}(\mathcal{C})$ has objects $"\varinjlim_{i \in I}" Y_i$ given by filtered diagrams $Y_i, i \in I$, of objects in $\mathcal{C}$, and morphisms

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}("\varinjlim_{i \in I}" Y_i, "\varinjlim_{j \in J}" Z_j) := \varprojlim_{j \in J} \varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(Y_i, Z_j).$$

Thus, $\mathrm{Ind}(\mathcal{C})$ is dual to the notion of a pro-category that we saw in Proposition 3.32. In fact, $\mathrm{Ind}(\mathcal{C}) \cong (\mathrm{Pro}(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}}$.

Lemma 4.20. *Let $\mathfrak{X}$ be a finitary topos, and let $\mathfrak{X}^{\mathrm{qcqs}}$ be its subcategory of qcqs objects. Then the functor*

$$\mathrm{Ind}(\mathfrak{X}^{\mathrm{qcqs}}) \to \mathfrak{X},\ "\varinjlim_{i \in I}" X_i \mapsto \varinjlim_{i \in I} X_i$$

is fully faithful.

Proof. This follows formally from Corollary 3.20 and the definition of an Ind-category □

We can now deduce from Lemma 4.19 a convenient description of quasi-separated condensed sets. We see in particular that κ-compactly generated Hausdorff spaces yield a large supply of quasi-separated condensed sets.

Lemma 4.21. (1) *The functor*

$$\Phi\colon \{qs\ condensed\ sets\} \to \mathrm{Ind}(\mathrm{CHaus}_\kappa),\ X \mapsto "\varinjlim"_{\underline{K} \subseteq X,\ K \in \mathrm{CHaus}_\kappa} K$$

is fully faithful with essential image given by Ind-systems with injective transition maps.

(2) *Let $X \in \mathrm{Top}^{\kappa-\mathrm{cg}}$. Then $\underline{X}$ is quasi-separated if and only if it is κ-weak Hausdorff, i.e., for each continuous map $f\colon S \to X$ with $S \in \mathrm{Prof}_\kappa$ the image $f(S) \subseteq X$ is compact Hausdorff for the subspace topologyd.* [12]

Proof. The first assertion follows formally from Lemma 4.20 and Lemma 4.19.

Let us show the second statement. Assume that $\underline{X}$ is quasi-separated. Then $X \cong \underline{X}(*)$ as X is assumed to be κ-compactly generated. Hence, it suffices to show that $T(*)_*$ is weak Hausdorff for any quasi-separated set T. As such we can write $T = \bigcup_{i \in I} \underline{K_i}$ as a filtered union for $K_i \in \mathrm{CHaus}_\kappa$. The functor $T \mapsto T(*)_*$ being a left adjoint, it suffices now to see that the colimit $Y = \varinjlim_{i \in I} K_i$ in topological spaces satisfies the statement in the lemma. Let $S \in \mathrm{Prof}_\kappa$ with a map $S \to Y$. each

[12]This notion is a κ-variant of the condition of being weak Hausdorff: the image of a continuous map from a compact Hausdorff space is compact Hausdorff for the subspace topology.

morphism $S \to \underline{X}$ with $S \in \mathrm{Prof}_\kappa$ is quasi-compact, and hence so is $Y := \mathrm{Im}(S \to \underline{X}) \to$. But this implies that $Y = \underline{K}$ for some closed subspace X by Lemma 4.17. Necessarily $K = Y(*) = \mathrm{Im}(S \to X)$ (by Remark 3.13). Assume the converse, and let $S_1, S_2 \in \mathrm{Prof}_\kappa$ with maps $S_1 \to X,\ S_2 \to X$. Set $K := \mathrm{Im}(S_1 \coprod S_2 \to X)$, which by assumption is a compact Hausdorff space for the subspace topology of X. This implies that

$$S_1 \times_X S_2 \cong S_1 \times_K S_2$$

is compact Hausdorff, and thus its associated κ-condensed set is qcqs (it also admits a surjection from a κ-profinite set as it embeds into $S_1 \times S_2$). □

Remark 4.22. The fully faithful embedding

$$\mathrm{Ind}(\mathrm{CHaus}_\kappa) \to \mathrm{CondSet}_\kappa$$

is likely not essentially surjective, even the condensed sets are generated by κ-profinite sets under colimits. The problem is the following: Let $X \in \mathrm{CondSet}_\kappa$. We can write X as a filtered colimit of quasi-compact objects (by considering images for morphisms $S \to X$ with $S \in \mathrm{Prof}_\kappa$), thus assume that X is quasi-compact, and take a surjection $S \to X$. Now, we'd like to write the equivalence relation $S \times_X S \to S \times S$ as a filtered colimit of *closed* equivalence relation (because their quotients will be in CHaus_κ by Lemma 3.21). We can again write the subspace $S \times_X S \times S \times S$ as a filtered colimit $\varinjlim_{i \in I} K_i$ of closed subspaces of $S \times S$ (because $S \times_X S$ is quasi-separated). However, the equivalence relation in $S \times S$ *generated* by K_i need not be closed.[13].

Remark 4.23. One has to be careful not to confuse an ind-system along injective maps with a union as a topological space. In general, it can happen that a compact Hausdorff space K is the union $K = \bigcup_{i \in I} K_i$ of closed subspaces, such that K carries the colimit topology. Indeed, let $f\colon \coprod_{i \in I} (\mathbb{N} \cup \{\infty\}) \to K := \{0,1\}^{\mathbb{N}}$, with the coprod taken over all continuous maps $\mathbb{N} \cup \{\infty\} \to K$. Because K is metrizable (it embeds as the Cantor set into $[0,1]$), the map f is a topological quotient map. For $i \in I$ let $K_i \subseteq K$ be the image of the corresponding map $\mathbb{N} \cup \{\infty\}$. Then the morphism $\bigcup_{i \in I} \underline{K_i} \to \underline{K}$ of quasi-separated condensed sets is *not* an isomorphism (otherwise K would be a finite union of K_i's). Note that the indexing set for such an example is necessarily uncountable, Lemma 4.6.

We can delve into this example: the subobject $\bigcup_{i \in I} \underline{K_i} \subseteq \underline{K}$ is not induced by a subspace of K, and hence is a genuinely "new" condensed set (in particular the quasi-compactness assumption in Lemma 4.17 is crucial).

Another property of $\underline{(-)}$ is that it does not preserve epimorphisms. This is actually a feature, and not a bug as the example $\mathbb{R}^{\mathrm{disc}} \to \mathbb{R}$ shows. However, this might even happen for quotient maps Remark 4.23.

We mention some positive results in this direction.

Lemma 4.24. *Let $f\colon Y \to X$ be a quotient map of κ-compactly generated spaces. Then $\underline{f}\colon \underline{Y} \to \underline{X}$ is an epimorphism if one of the following conditions are satisfied:*

1. *X is κ-weak Hausdorff, i.e., $\underline{X}$ is quasi-separated, and each compact Hausdorff subspace $K \subseteq X$ with $K \in \mathrm{CHaus}_\kappa$ lies in the image $f(L)$ of a compact Hausdorff subspace $L \subseteq Y$, which lies in CHaus_κ.*
2. *κ is a strong limit cardinal, Y, X are locally compact Hausdorff spaces with $|Y|, |X| < \kappa$, and f is proper or open.*

In Lemma 4.33 we establish a version of (2) for $\kappa = \omega_1$.

Proof. (1): This is almost directly the definition of an epimorphism of condensed sets: let $S \in \mathrm{Prof}_\kappa$. We need to see that $\underline{Y} \times_{\underline{X}} S \to S$ is an epimorphism. But $K := \mathrm{Im}(S \to \underline{X}) \subseteq \underline{X}$ is a compact Hausdorff subspace of X (using that X is quasi-separated), and hence lies in the image of some $L \subseteq Y$, $L \in \mathrm{CHaus}_\kappa$. But this means that

$$\underline{Y} \times_{\underline{X}} S \cong \underline{L} \times_{\underline{f(L)}} S \cong \underline{L \times_{f(L)} S}$$

lies in CHaus_κ, and surjects onto S. But this implies that the morphism to S is an epimorphism as desired.

(2): Because κ is a strong limit cardinal a compact Hausdorff space K lies in CHaus_κ if and only if $|K| < \kappa$. This implies that each compact subspace of X or Y lies in CHaus_κ. If f is proper, then by definition for each compact subspace $K \subseteq X$ the inverse image $f^{-1}(K) \subseteq Y$ is compact, and hence (1) is satisfied. Assume that f is open, and let $K \subseteq X$ be compact. For each $y \in f^{-1}(K)$, pick some compact neighborhood L_y of y in Y. As f is open, $f(L_y)$ is a neighborhood of $f(y)$.

[13] It would be nice to have a concrete example, where this happens.

This implies that K is covered by finitely many $f(L_y)$'s, and thus lies in the image of a finite union of the L_y's. □

Any topological space admits a maximal Hausdorff quotient.[14] Similarly, each condensed set admit a maximal quasi-separated quotient.

Lemma 4.25. *The inclusion* $\{qs\ \kappa\text{-condensed sets}\} \to \mathrm{CondSet}_\kappa$ *admits a left adjoint* $X \mapsto X^{\mathrm{qs}}$*, which commutes with finite products. In particular, limits of quasi-separated* κ*-condensed sets are quasi-separated.*

Proof. Let $X \in \mathrm{CondSet}_\kappa$, and choose a surjection $X' = \coprod_{i\in I} S_i \to X$ with $S_i \in \mathrm{Prof}_\kappa$. Let $R := X' \times_X X' \subseteq X' \times X'$ be the induced equivalence relation. Note that for any map $X \to Y$ with Y quasi-separated, the induced map $X' \to Y$ has the property that $X' \times_Y X' \subseteq X' \times X'$ is a quasi-compact injection (hence identifies with the inclusion of a closed subspace by Lemma 4.17), and it contains R (because $X' \to Y$ factors over X). Let $\overline{R} \subseteq X' \times X'$ be the minimal *closed* equivalence relation containing R. Note that $\overline{R}$ exists because any intersection of closed equivalence relations in $X' \times X'$ is again a closed equivalence relation. In particular, we can set $X^{\mathrm{qs}} := X'/\overline{R}$ as it satisfies the required universal property.

Let us now check that $X \mapsto X^{\mathrm{qs}}$ commutes with finite products. It suffices to see that if $R \subseteq X \times X$ and $R' \subseteq X' \times X'$ are two equivalence relations on quasi-separated condensed sets X, X', then the minimal closed equivalence relation $\overline{R \times R'}$ on $X \times X'$, containing $R \times R'$ is given by $\overline{R} \times \overline{R'}$. Note first that for $x' \in X'$ fixed, $\overline{R \times R'}$ needs to contain $\overline{R} \times \{(x', x')\} \subseteq X \times X \times X' \times X'$. Similarly, for fixed $x \in X$, we have $\{(x,x)\} \times \overline{R'} \subseteq \overline{R \times R'}$. Now, assume that $(x_1, x_1'), (x_2, x_2') \in X \times X'$ are two elements, such that x_1, x_2 are equivalent through $\overline{R}$, and x_1', x_2' through $\overline{R'}$. Then (x_1, x_1') is $\overline{R \times R'}$-equivalent to (x_2, x_1') (because $\overline{R} \times \{(x_1', x_1')\}) \subseteq \overline{R \times R'}$) and (x_2, x_1') is $\overline{R \times R'}$-equivalent to (x_2, x_2'). This implies that $\overline{R} \times \overline{R'} \subseteq \overline{R \times R'}$ as desired. □

Example 4.26. The equivalence relation $R \subseteq \mathbb{R} \times \mathbb{R}$ given by the action of $\mathbb{Q}$ by addition, i.e., $R = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x - y \in \mathbb{Q}\}$, has closure $\mathbb{R} \times \mathbb{R}$, and hence the quasi-separated quotient of $\underline{\mathbb{R}}/\underline{\mathbb{Q}}$ is $\{*\}$.

4.4. **Sequential spaces and light condensed sets.** We now assume that $\kappa = \omega_1$ is the first uncountable cardinal, so that ω_1-condensed sets are by definition light condensed sets. We will therefore use the notation

$$\mathrm{Prof}^{\mathrm{light}} := \mathrm{Prof}_{\omega_1},\ \mathrm{CondSet}^{\mathrm{light}} := \mathrm{CondSet}_{\omega_1}.$$

As the case of light condensed sets is the most relevant case, we spent some time to discuss this case.

Let us first discuss ω_1-compactly generated spaces.

Lemma 4.27. *Let X be a topological space. The following are equivalent:*

1. *X is ω_1-compactly generated, Definition 4.2,*
2. *X is a sequential space, i.e., X is a topological quotient $Y \twoheadrightarrow X$ of a first countable space Y,*
3. *X is a quotient of a metrizable space,*
4. *X is a quotient of $\coprod_{i\in I}(\mathbb{N} \cup \{\infty\})$ for some set I.*

Note that an ω_1-compactly generated space will be κ-compactly generated for each uncountable cardinal κ.

Proof. Assume that X is ω_1-compactly generated. By definition, this means that X admits a quotient map $\coprod_{i\in I} S_i \to X$ for some set I and $S_i \in \mathrm{Prof}^{\mathrm{light}}$, $i \in I$. By Lemma 4.28 we may replace S_i by the Cantor set $\{0,1\}^{\mathbb{N}}$. But the Cantor set is metrizable (as is any *countable* product of metrizable spaces[15].) Now, each first countable space X is the topological quotient along the natural map

$$\coprod_{\mathbb{N}\cup\{\infty\} \to X \text{ cts.}} \mathbb{N} \cup \{\infty\} \to X,$$

because a map $X \to Y$ to a topological space is continuous if and only if it maps a convergent sequence $x_n \to x,\ n \to \infty$ in X to the convergent sequence $f(x_n) \to f(x),\ n \in \mathbb{N}$. We note that $\mathbb{N} \cup \{\infty\} \cong \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$ is clearly metrizable. Hence, we are left with proving that a sequential space is ω_1-compactly generated. Thus assume that X is sequential. Each quotient

[14]This follows from the fact that limits of Hausdorff spaces are Hausdorff.

[15]If $X_n,\ n \in \mathbb{N}$ are metrizable with metrics d_n, then $\prod_{n\in\mathbb{N}} X_n$ is metrizable with metric $d((x_n),(y_n)) := \sum_{n\in\mathbb{N}} \frac{d_n(x_n,y_n)}{2^n(1+d_n(x_n,y_n))}$

of a κ-compactly generated space will be κ-compactly generated, so we may assume that X is first-countable. Let $V \subseteq X$ be a subset, such that for all $S \in \mathrm{Prof}^{\mathrm{light}}$ mapping to X, the preimage of V in S is closed. We need to see that V is closed. Take $x \in X$ in the closure of V, and choose a sequence $U_1 \supseteq U_2 \supseteq \ldots$ of cofinal open neighborhoods of x (using that X is first-countable). As $V \cap U_n$ is nonempty for each n, we can choose some $x_n \in V \cap U_n$ a produce a continuous map $\mathbb{N} \cup \{\infty\} \to X,\ n \mapsto x_n,\ \infty \mapsto x$. Now $\mathbb{N} \cup \{\infty\} \in \mathrm{Prof}_\kappa$ (this uses that κ is uncountable), and thus the inverse image of V is closed in $\mathbb{N} \cup \{\infty\}$. As it contains $\mathbb{N}$, it must be $\mathbb{N} \cup \{\infty\}$, and thus $x \in V$ and V is closed. □

Lemma 4.28. *Let $S \in \mathrm{Prof}^{\mathrm{light}}$. Then S admits a surjection $\{0,1\}^{\mathbb{N}} \to S$ from the Cantor set $\{0,1\}^{\mathbb{N}}$.*

Proof. Write $S = \prod_{n \in \mathbb{N}} S_n$ with S_n finite and discrete. There exists some $m_0 \geq 1$ and a surjection $0,1^{m_0} \to S_0$. Then there exists some $m_1 > m_0$ and a commutative diagram

$$\begin{array}{ccc} \{0,1\}^{m_1} & \longrightarrow & S_1 \\ {\scriptstyle \text{proj.}}\downarrow & & \downarrow \\ \{0,1\}^{m_0} & \longrightarrow & S_0. \end{array}$$

Continuing we find a surjection $\{0,1\}^{\mathbb{N}} \cong \varprojlim_{i=0}^{\infty} \{0,1\}^{m}_{i} \to S$. □

Definition 4.29. We denote by $\mathrm{Top}^{\mathrm{met}} \subseteq \mathrm{Top}^{\mathrm{seq}} \subseteq \mathrm{Top}$ the full subcategories of metrizable resp. sequential spaces.

We warn the reader that the functor $\underline{(-)}$ does not in general preserve quotients of metrizable spaces.

Example 4.30. Any topological quotient $f\colon \coprod_{i \in I} \mathbb{N} \cup \{\infty\} \to C := \{0,1\}^{\mathbb{N}}$ to the Cantor set does not induce a quotient on (light) condensed sets. Otherwise, there would exist a surjection $S \to C$ with S profinite and a map $g\colon S \to \coprod_{i \in I} \mathbb{N} \cup \{\infty\}$ lifting f. But the map g has to factor through finitely many summands (because S is compact), and hence C would be countable.

We can identify qcqs light condensed sheaves conveniently by metrizable compact Hausdorff spaces.

Lemma 4.31. *The following categories are equivalent:*

1. *metrizable compact Hausdorff spaces,*
2. *second-countable compact Hausdorff spaces,*
3. *compact Hausdorff spaces, which admit a surjection from a light profinite set,*
4. *qcqs light condensed sets.*

Moreover, if $K' \to K$ is a surjection of metrizable compact Hausdorff spaces, then $\underline{K'} \to \underline{K}$ is an epimorphism of light condensed sets.

Proof. The equivalence of the first two points is a special case of Urysohn's metrizability theorem[16], and rests on Urysohn's lemma, which in our case implies that points in the compact Hausdorff space K can be separated by maps to the interval $[0,1]$: Given a point $x \in K$ and a closed subset $Z \subseteq K$ in the compact Hausdorff space K with $x \notin Z$, there exists a continuous function $f\colon K \to [0,1]$ with $f_{|Z} = 1$ and $f(x) = 0$. If K is second-countable, then a countable collection of continuous maps $K \to [0,1]$ as before will define an embedding (equivalently an injection) $K \to [0,1]^{\mathbb{N}}$ into the Hilbert cube. But the latter is metrizable, so K is metrizable. Moreover, the Hilbert cube admits a surjection from light profinite set, by taking the product of the surjections

$$\{0,1\}^{\mathbb{N}} \to [0,1],\ (a_n)_n \mapsto \sum_{n=1}^{\infty} a_n 2^{-n}.$$

If a compact Hausdorff space admits a surjection from a light profinite set, then it must be second-countable by Lemma 4.33.

The equivalence to qcqs light condensed sets follows from Lemma 4.18. □

Example 4.32. The following example of a first countable (hence sequential), but not metrizable compact Hausdorff space K was explained to us by O. Gabber. For a finite set $J \subseteq [0,1]$ consider the subspace $X_J = [0,1] \times \{0\} \cup \bigcup_{j \in J} \{j\} \times [0,1] \subseteq [0,1] \times [0,1]$. For $J' \subseteq J$ we get a natural transition map $X_J \to X_{J'}$ which is the identity on $X_{J'}$, but which is the projection to the first

[16] Any second-countable regular Hausdorff space is metrizable.

factor on $\{j\} \times [0,1]$ for $j \in J \setminus J'$. As a set, the inverse limit $X := \varprojlim_{J \subseteq [0,1]}$ is in bijection to $[0,1] \times [0,1]$, but the topology is different. In fact, each $\{j\} \times [0,1)$ for $j \in [0,1]$ is open, and from here one sees that X cannot be second countable. On the other hand, each point in X has a countable neighborhood basis by using the usual balls in $[0,1] \times [0,1]$ of shrinking radii for points in $[0,1] \times \{0\}$, and the usual neighborhoods in $\{j\} \times [0,1)$ for $j \in [0,1]$.

We used the following lemma.

Lemma 4.33. *Let $f\colon S \to K$ be a surjection from a metrizable compact Hausdorff space, e.g., a light profinite set, to a compact Hausdorff space. Then K is metrizable, and hence second-countable.*

Proof. Let $d\colon S \times S \to \mathbb{R}_{\geq 0}$ be a metric, and for $x, y \in K$ set

$$e(x,y) := \inf\{d(s_1,s_2)+\ldots+d(s_{n-1},s_n) \mid x = f(s_1), f(s_2) = f(s_3), \ldots, f(s_{n-1}) = f(s_n), f(s_n) = y\}.$$

One easily checks that $e\colon K \times K \to \mathbb{R}_{\geq 0}$ defines a pseudo-metric. We show that it is even a metric, i.e., $e(x,y) = 0$ forces $x = y$. Thus, assume that $e(x,y) = 0$ and $x \neq y$. As K is Hausdorff, there exist disjoint open neighborhoods $U, V \subseteq K$ of x, y. But then for some $\varepsilon > 0$ the inverse images $f^{-1}(U), f^{-1}(V)$ contain some ε-neighborhood of $f^{-1}(x)$ resp. $f^{-1}(y)$, i.e., the union of the ε-neighborhoods of each point in $f^{-1}(x)$ resp. $f^{-1}(y)$. This implies that $e(x,y)$ must be bigger than ε, contradiction.

Given that e is a metric, it is clear that f is continuous when K is equipped with the topology for e. However, as f is a quotient map, this forces the given topology on K to agree with the metric topology for e. □

We get the following description of quasi-separated light condensed sets.

Corollary 4.34. *The category of quasi-separated light condensed sets is equivalent to the full subcategory* $\mathrm{Ind}_{\mathrm{inj}}(\mathrm{CHaus}^{\mathrm{met}}) \subseteq \mathrm{Ind}(\mathrm{CHaus}^{\mathrm{met}})$ *consisting of filtered systems with injective transition maps.*

Proof. This follows from Lemma 4.31 and Lemma 4.21. □

Sequential spaces exist in abundance.

Example 4.35. The following are examples of sequential spaces:

(1) Metrizable topological spaces,
(2) CW-complexes (being a quotient of a disjoint union of closed balls).

We note cases, where surjections of sequential topological spaces map to surjections of condensed sets, in addition to Lemma 4.24.

Lemma 4.36. *Let $f\colon Y \to X$ be a surjective maps of metrizable topological spaces. Assume that f is proper or open. Then $\underline{f}\colon \underline{Y} \to \underline{X}$ is surjective.*

Proof. Note that each compact subspace of Y or X is metrizable, and hence the assertion follows from the proof Lemma 4.24. □

4.5. **Interlude: morphisms of topoi.** In this section we want to discuss the notion of a "morphism between topoi". The basic example for a topos is the category of sheaves on a topological space (Example 3.2). Assume that $f\colon Y\to X$ is a continuous map of topological spaces. Then we obtain a pullback functor

$$f^{-1}_{\mathrm{sites}}\colon \mathrm{Ouv}(X)\to \mathrm{Ouv}(Y),\ (Z\to X)\mapsto (Z\times_X Y\to Y)$$

on associated sites, which preserves finite limits and maps a covering $\{Z_i\to Z\}_{i\in I}$ to the covering $\{Z_i\times_X Y\to Y\}_{i\in I}$ (in anticipation of the general case we write fiber products and not the intersection of open subsets). This implies that the functor

$$f^P_*\colon \mathrm{PSh}(\mathrm{Ouv}(Y))\to \mathrm{PSh}(\mathrm{Ouv}(X)),\ \mathcal{F}\mapsto \mathcal{F}\circ f^{-1}_{\mathrm{sites}}$$

induces a functor

$$f_*\colon \mathrm{Sh}(Y)\to \mathrm{Sh}(X).$$

In any discussion for sheaves on topological spaces, one quickly proves the existence of a left adjoint f^{-1}, *which is exact, i.e., commutes with finite limits.* This commutation with finite limits is important, e.g., it implies that f^{-1} preserves sheaves of abelian groups/ring/... and defines an exact functor for such. Motivated by this we make the following definition.

Definition 4.37. A morphism $f\colon \mathfrak{Y}\to\mathfrak{X}$ of topoi is a functor $f_*\colon \mathfrak{Y}\to\mathfrak{X}$, which admits an exact left adjoint $f^{-1}\colon \mathfrak{X}\to\mathfrak{Y}$. If $g\colon \mathfrak{Z}\to\mathfrak{Y},\ f\colon\mathfrak{Y}\to\mathfrak{X}$ are morphisms of topoi, then their composition $f\circ g$ is given by the functor $f_*\circ g_*$, whose left adjoint $g^{-1}\circ f^{-1}$ is indeed exact.

As equality of functors is badly behaved for categories it is often better to also keep track of the natural transformations $\eta\colon f_*\to f'_*$ for two morphisms $f,f'\colon\mathfrak{Y}\to\mathfrak{X}$ of topoi. Thus topoi form naturally a 2-category. We will not enter this discussion in this lecture as it will not be important.

The next lemma is a basic source for constructing morphisms of topoi.

Lemma 4.38. *Let $u\colon\mathcal{C}\to\mathcal{D}$ be a functor between small categories and define the functor*

$$u^{\mathrm{PSh}}_*\colon \mathrm{PSh}(\mathcal{D})\to\mathrm{PSh}(\mathcal{C}),\ F\mapsto F\circ u.$$

(1) *The functor u^{PSh}_* admits a left adjoint $u^{-1}_{\mathrm{PSh}}\colon \mathrm{PSh}(\mathcal{C})\to\mathrm{PSh}(\mathcal{D})$.*
(2) *If $G\in\mathrm{PSh}(\mathcal{C})$, then $u^{-1}_{\mathrm{PSh}}(G)$ is the "pointwise left Kan extension" of G along u, i.e.,*

$$(Z\in\mathcal{D})\mapsto \varinjlim_{\{Y\in\mathcal{C},\ Z\to u(Y)\}^{\mathrm{op}}} G(Y).$$

(3) *If $Y\in\mathcal{C}$ with contravariant Hom-functor $h_Y:=\mathrm{Hom}_{\mathcal{C}}(-,Y)\in\mathrm{PSh}(\mathcal{C})$, then*

$$u^{-1}_{\mathrm{PSh}}(h_Y)=h_{u(Y)}.$$

(4) *Assume that $\mathcal{C}$ has finite limits, and u preserves these. Then u^{-1}_{PSh} is exact. In particular, u^{PSh}_* defines a morphism $u^{\mathrm{PSh}}\colon\mathrm{PSh}(\mathcal{D})\to\mathrm{PSh}(\mathcal{C})$ of topoi.*

Note that the smallness of $\mathcal{C},\mathcal{D}$ implies here that the colimits in (2) exists for any $Z\in\mathcal{D}$. Before starting the proof let us consider the example

$$u=f^{-1}\colon\mathrm{Ouv}(X)\to\mathrm{Ouv}(Y)$$

for a continuous map $f\colon Y\to X$ of topological spaces from before. Then

$$u^{-1}_{\mathrm{PSh}}(\mathcal{G})(U)=\varinjlim_{V\subseteq S\text{ open },\ U\subseteq f^{-1}(V)}\mathcal{G}(V).$$

Proof. We can define u^{-1}_{PSh} by the formula in (2). Then it is straightforward to check that u^{-1}_{PSh} is left adjoint to u^{PSh}_* by identifying elements in $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{D})}(u^{-1}_{\mathrm{PSh}}(\mathcal{G}),\mathcal{F})$ and $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\mathcal{G},u^{\mathrm{PSh}}_*(\mathcal{F}))$ with systems of maps $\varphi_{Y,Z,f}\colon\mathcal{G}(Y)\to\mathcal{F}(Z)$ for any $Y\in\mathcal{C},Z\in\mathcal{D}$ and morphism $f\colon Z\to u(Y)$, which are compatible for varying the data Y,Z,f.

Let us show (3). Take $Y\in\mathcal{C}$ and $\mathcal{G}\in\mathrm{PSh}(\mathcal{D})$. Then

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{PSh}(\mathcal{D})}(u^{-1}_{\mathrm{PSh}}(h_Y),\mathcal{G})\\
\overset{(2)}{=}\ &\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_Y,u^{\mathrm{PSh}}_*(\mathcal{G}))\\
\overset{Yoneda}{=}\ &u_*(\mathcal{G})(Y)\\
=\ &\mathcal{G}(u(Y))\\
\overset{Yoneda}{=}\ &\mathrm{Hom}_{\mathrm{PSh}(\mathcal{D})}(h_{u(Y)},\mathcal{G}).
\end{aligned}$$

By a third application of the Yoneda lemma we can conclude that $u_{\mathrm{PSh}}^{-1}(h_Y)$ and $h_{u(Y)}$ are naturally isomorphic. Finally let us prove (4). From the pointwise construction of u_{PSh}^{-1} it suffices to check that for each $Z \in \mathcal{D}$ the category

$$I_Z := \{Y \in \mathcal{C},\ Z \to u(Y)\}^{\mathrm{op}}$$

is filtered. But as $\mathcal{C}$ has finite limits and u commutes with these, it follows easily that I_Z has finite colimits. But any category with finite colimits is filtered. □

Definition 4.39. Let $\mathcal{C}, \mathcal{D}$ be sites and let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor. Assume that

(1) the category $\mathcal{C}$ has finite limits and u commutes with these.
(2) if $\{Y_i \to Y\}_{i\in I}$ is a covering in $\mathcal{C}$, then $\{u(Y_i) \to u(Y)\}_{i\in I}$ is a covering in $\mathcal{D}$.

Then we set

$$f_*\colon \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C}),\ \mathcal{F} \mapsto u_*^{\mathrm{PSh}}(\mathcal{F}) = \mathcal{F} \circ u$$

and

$$f^{-1}\colon \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{D}),\ \mathcal{G} \mapsto (u_{\mathrm{PSh}}^{-1}(\mathcal{G}))^\sharp,$$

and call the associated morphism $f\colon \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C})$ of topoi the morphism associated with u.

Example 4.40. Let $\kappa < \kappa'$ be uncountable cardinals. Then the inclusion $u\colon \mathcal{C} := \mathrm{Prof}_\kappa \to \mathcal{D} := \mathrm{Prof}_{\kappa'}$ satisfies the assumptions of Definition 4.39 because the fiber product of κ-profinite sets is again κ-profinite (this is Lemma 3.56). Hence, we get an exact, left adjoint functor

$$\mathrm{CondSet}_\kappa \to \mathrm{CondSet}_{\kappa'}$$

to the restriction functor $\mathrm{CondSet}_{\kappa'} \to \mathrm{CondSet}_\kappa,\ \mathcal{F} \mapsto \mathcal{F}_{|\mathrm{Prof}_\kappa^{\mathrm{op}}}$.

4.6. **Interlude: Dependence on κ.** By definition, κ-condensed sets depend on the chosen cardinal κ. In this section, we now want to adress the (subtle) question, what happens if one enlarges κ. We note that for most applications it is sufficient to take $\kappa = \omega_1$, so that the discussion below is most often not very relevant.

Definition 4.41. A cardinal κ is regular if for any union $S = \bigcup_{i\in I} S_i$ of sets with $|S_i|, |I| < \kappa$ also $|S| < \kappa$.

For example, ω and ω_1 are regular (more generally, each infinite successor cardinal is regular), but $\beth_\omega = \bigcup_{n<\omega} \beth_n$ is not.[17]

Proposition 4.42. *Let $\kappa < \kappa'$ be uncountable cardinals. Assume that κ is regular. Then the pullback functor*

$$\nu^{-1} := \nu^{-1}_{\kappa,\kappa'} \colon \mathrm{CondSet}_\kappa \to \mathrm{CondSet}_{\kappa'}$$

from Example 4.40 is fully faithful, and its essential image are those $\mathcal{G} \in \mathrm{CondSet}_{\kappa'}$ such that for any $T \in \mathrm{Prof}_{\kappa'}$, which is written as a cofiltered limit $T = \varprojlim_{i\in I} T_i$ of $T_i \in \mathrm{Prof}_\kappa$, the natural morphism $\varinjlim_{i\in I} \mathcal{G}(T_i) \to \mathcal{G}(T)$ is bijective.

Proof. Let $u\colon \mathrm{Prof}_\kappa \to \mathrm{Prof}_{\kappa'}$ be the inclusion, and let $\mu = u^{-1}_{\mathrm{PSh}} \colon \mathrm{PSh}(\mathrm{Prof}_\kappa) \to \mathrm{PSh}(\mathrm{Prof}_{\kappa'})$ be the left adjoint from Lemma 4.38 to the restriction functor $u_*^{\mathrm{PSh}} \colon \mathrm{PSh}(\mathrm{Prof}_{\kappa'}) \to \mathrm{PSh}(\mathrm{Prof}_\kappa)$, $\mathcal{G} \mapsto \mathcal{G} \circ u$. Let $\nu_* \colon \mathrm{CondSet}_{\kappa'} \to \mathrm{CondSet}_\kappa$ be the restriction of u_*^{PSh} to condensed sets, i.e., sheaves. Let $\mathcal{F} \in \mathrm{CondSet}_\kappa$. Recall that $\nu^{-1}(\mathcal{F})$ is the sheafification of $u^{-1}_{\mathrm{PSh}}(\mathcal{F})$. It suffices to see that the natural morphism

$$\mathcal{F} \to \nu_*\nu^{-1}(\mathcal{F})$$

is an isomorphism. Evaluating on $S \in \mathrm{Prof}_\kappa$, we see that it suffices to see that $u^{-1}_{\mathrm{PSh}}(\mathcal{F})$ is already a sheaf on $\mathrm{Prof}_{\kappa'}$. Let $T \in \mathrm{Prof}_{\kappa'}$. Recall from Lemma 4.38 that

$$u^{-1}_{\mathrm{PSh}}(\mathcal{F})(T) = \varinjlim_{T\to S,\ S\in\mathrm{Prof}_\kappa} \mathcal{F}(S).$$

Let $T = \varprojlim_{j\in J} S_j \in \mathrm{Prof}_{\kappa'}$ with $S_j \in \mathrm{Prof}_\kappa$ and J κ-cofiltered.[18] We claim that

$$u^{-1}_{\mathrm{PSh}}(\mathcal{F})(T) = \varinjlim_{j\in J} \mathcal{F}(S_j),$$

i.e., the above colimit may be replaced by a suitable over colimit. To prove this claim it suffices to see that any continuous morphism $T \to S$ with $S \in \mathrm{Prof}_\kappa$ factors over some $T \to S_j$. On boolean algebras this means that the morphism $h\colon \mathrm{Cont}(S, \mathbb{F}_2) \to \mathrm{Cont}(T, \mathbb{F}_2)$ factors over some $\mathrm{Cont}(S_j, \mathbb{F}_2)$. Now, the image of h will be contained in the image of some $\mathrm{Cont}(S_j, \mathbb{F}_2)$ because $\mathrm{Cont}(S, \mathbb{F}_2) < \kappa$, and J is κ-filtered. Similarly, the at most κ-many relations of $< \kappa$-many generators of $\mathrm{Cont}(S, \mathbb{F}_2)$ will be realized already in some $\mathrm{Cont}(S_j, \mathbb{F}_2)$. This implies the claim. Now, κ being regular implies that we can write each $T \in \mathrm{Prof}_{\kappa'}$ as a κ-cofiltered inverse limits of $S_j \in \mathrm{Prof}_\kappa$. [19] Indeed, we can write $\mathrm{Cont}(T, \mathbb{F}_2)$ as the filtered union of all of its Boolean subalgebras of cardinality $< \kappa$, and because κ is regular, this union is κ-filtered. Let now $\mathcal{U} := \{T_i \to T\}_{i\in I}$ be a covering in $\mathrm{Prof}_{\kappa'}$. By applying Lemma 4.43 below to $\coprod_{i\in I} T_i \to T$ we can write $T_i \to T$ as a cofiltered inverse limit of coverings $\mathcal{U}_j := \{S_{ij} \to S_j\}_{i\in I}$, $j \in J$, with $S_j, S_{ij} \in \mathrm{Prof}_\kappa$. As κ is regular, we may by the same reasoning as above even assume that this inverse limit is κ-cofiltered. But combining the previous assertions, we see that (in the notation of Section 3.1)

$$\Gamma(T, u^{-1}_{\mathrm{PSh}}\mathcal{F}) = \varinjlim_{j\in J} \Gamma(S_j, \mathcal{F}) = \varinjlim_{j\in J} \Gamma(\mathcal{U}_j, \mathcal{F}) = \Gamma(\mathcal{U}, u^{-1}_{\mathrm{PSh}}\mathcal{F})$$

using in the last step that filtered colimits commute with finite inverse limits. The description of the essential image expresses the statement the counit
$nu^{-1}\nu_*\mathcal{G} \to \mathcal{G}$ is an isomorphism, and hence this describes the essential image. This finishes the proof. □

We needed the following lemma when comparing κ-condensed sets with κ'-condensed sets for $\kappa < \kappa'$.

[17] The existence of arbitrary large regular *and* strong limit cardinals cannot be proved within ZFC, so we avoid them in this course.

[18] A κ-filtered colimit is a colimit over a κ-filtered category I and a category I is κ-filtered if any diagram $f\colon J \to I$ with J having $< \kappa$ arrows, has a a cocone, i.e., there exists some $i \in I$ with compatible morphisms $f(j) \to i$ for all $j \in J$. The dual notion to being κ-filtered is beign κ-cofiltered.

[19] In fact, $\mathrm{Prof}_{\kappa'}$ is the $\mathrm{Pro}_{\kappa-\mathrm{filterd}}$-category of Prof_κ, where the κ-filtered Pro-category considers only κ-cofiltered pro-systems.

Lemma 4.43. *Let $\kappa < \kappa'$ be uncountable cardinals. Let $S' \to S$ be a surjection of κ'-profinite set. Then $S' \to S$ is a cofiltered inverse limit of surjections $T_i \to S_i$ of κ-profinite sets.*

Proof. Set $A := \mathrm{Cont}(S, \mathbb{F}_2) \to B := \mathrm{Cont}(S', \mathbb{F}_2)$ as the associated morphism of Boolean algebras (Proposition 3.32). Note that surjectivity of $S' \to S$ is equivalent to injectivity of $A \to B$. Let $C \subseteq B$ be a subset with $|C| < \kappa$. Then the Boolean algebra generated by C in B has size $< \kappa$ (because it is a quotient of $\mathbb{F}_2[X_c | c \in C]/(X_c^2 - X_c)$). Hence, we can write $B = \varinjlim\limits_{i \in I} B_i$ as a filtered colimit of Boolean subalgebras B_i with $|B_i| < \kappa$. Set A_i as the inverse image of B_i in A. Then $A_i \to B_i$ is injective, and hence $|A_i| < \kappa$. Thus, the morphism $T_i := \mathrm{Spec}(B_i) \to S_i := \mathrm{Spec}(A_i)$ is a surjection of κ-profinite sets. Because, $A \to B$ is the filtered colimit of the morphisms $A_i \to B_i$, the morphism $S' \to S$ is the filtered colimit of the $T_i \to S_i$. □

Remark 4.44. Generalizing the assertion that filtered colimits (=ω-filtered colimits) of sets commute with finite limits, one can show that κ-filtered colimits commute with κ-small limits.[20] As a corollary from the proof of Proposition 4.42 one sees therefore that the functor $\nu_{\kappa,\kappa'}^{-1}$ commutes with κ-small limits if κ is regular.

We make a warning.

Remark 4.45. If $\kappa < \kappa'$ with pullback $\nu^{-1} \colon \mathrm{CondSet}_\kappa \to \mathrm{CondSet}_{\kappa'}$, then $\nu_* \circ \underline{(-)}_{\kappa'} = \underline{(-)}_\kappa$ by definition, but $\nu^{-1} \circ \underline{(-)}_\kappa \ncong \underline{(-)}_{\kappa'}$, i.e., the realization of a topological space X as a condensed set depend on κ! Namely, let $X = \{\eta, s\}$ be the Sierpinski space, i.e., the opens of X are $\emptyset, \{\eta\}, \{\eta, s\}$. Then

$$\mathrm{Hom}_{\mathrm{cts}}(S, X) \cong \{Z \subset S \text{ closed}\},\ \varphi \mapsto \varphi^{-1}(s)$$

is a bijection for any profinite set S (even any topological space). If $\nu^{-1}(\underline{X}_\kappa) = \underline{X}_{\kappa'}$, then for any $S \in \mathrm{Prof}_{\kappa'}$, written as a cofiltered limit $S = \varprojlim_{i \in I} S_i$ with $S_i \in \mathrm{Prof}_\kappa$, then each closed subset Z of S is the inverse image of a closed subset of some S_i. But on Boolean algebras, this implies that the kernel of $\mathrm{Cont}(S, \mathbb{F}_2) \to \mathrm{Cont}(Z, \mathbb{F}_2)$ is generated by $< \kappa$ many elements. But this is not necessarily possible: Take α a sufficiently large ordinal, and set $S = \alpha + 1$. Then the closed point $Z := S \setminus \alpha$ gives a counter-example.

There exist worse counterexamples: if K is a sequential compact Hausdorff space, which is not metrizable, cf. Example 4.32, then $\nu^{-1}\underline{K}_{\omega_1} \neq \underline{K}_\kappa$ if $2^{2^{|K|}} < \kappa$. Indeed, $\nu^{-1}\underline{K}_{\omega_1}$ is given by the non-quasi-compact, quasi-separated κ-condensed set given by the filtered colimit $\varinjlim_{i \in I} \underline{K_i}_\kappa$ with I running through all metrizable compact subspaces $K_i \to K$, while $\underline{K}_\kappa$ is qcqs as it admits the surjection from the κ-profinite set $\beta(K^{\mathrm{disc}})$.

To solve this issue it helps to require that there exists (not too many) *closed* subsets separating points.

Lemma 4.46. *Let $\kappa < \kappa'$ be uncountable cardinals, and assume that κ is regular. Let X be a κ-compactly generated space such that for each quasi-compact subset $Z \subseteq X$ there exists closed subsets $K_j, j \in J$, with $|J| < \kappa$, such that for each $x, y \in Z, x \neq y$ there exists $j, j' \in J$ with $K_j \cap K_{j'} = \emptyset$ and $x \in K_j,\ y \in K_{j'}$. Then $\nu^{-1}\underline{X}_\kappa = \underline{X}_{\kappa'}$.*

The proof is an adaption of [Sch, Proposition 2.15].

Proof. By Proposition 4.42 and Remark 4.44 it suffices to see that the natural map

$$\Phi \colon \varinjlim_{i \in I} \mathrm{Hom}_{\mathrm{cts}}(S_i, X) \to \mathrm{Hom}_{\mathrm{cts}}(S, X)$$

is bijective for any $S \in \mathrm{Prof}_{\kappa'}$, which is written as the κ-cofiltered limit $S = \varprojlim_{i \in I} S_i$ of all maps $S \to S_i$ with $S_i \in \mathrm{Prof}_\kappa$. Note that within the cofiltered system of all maps $S \to S_i$ to κ-profinite sets, the surjections are cofinal because the image will be κ-profinite (by Lemma 3.56). This implies that Φ is injective. Let $f \colon S \to X$ be a continuous map. It suffices to see that there exists some i, such that

$$S \times_{S_i} S \subseteq S \times_X S \subseteq S \times S.$$

Note that the image $Z := f(X) \subseteq X$ is quasi-compact (but not necessarily Hausdorff). By hypothesis, there exist $< \kappa$-many closed subsets $K_j, j \in J$, such that for any $x_1, x_2 \in Z$ there exist $j_1, j_2 \in J$ with $x_k \in K_{j_k}$ and $K_{j_1} \cap K_{j_2} = \emptyset$. Thus, the sets $T_{j,j'} := f^{-1}(K_j) \times f^{-1}(K_{j'}) \subseteq S \times S$ for $j, j' \in J$ with $K_j \cap K_{j'} \neq \emptyset$ cover $S \times S \setminus S \times_X S$. Set $T_{j,j',i} := S \times_{S_i} S \cap T_{j,j'}$. Then the cofiltered limit over i of all the compact subspaces $T_{j,j',i}$ of $S \times S$ is empty (because $T_{j,j'}$ does not meet the diagonal of S). But this implies that for any pair $j, j' \in J$ as above, there exists some

[20] A limit is κ-small if its indexing category has $< \kappa$ arrows.

$i_{j,j'}$ with $T_{j,j',i} = \emptyset$. As I is κ-filtered and $|J| < \kappa$, this implies that we can find a single $i \in I$ with $S \times_{S_i} S \cap (S \times S \setminus S \times_X S) = \emptyset$, i.e., $S \times_{S_i} S \subseteq S \times_X S$ as desired. □

Let us give some cases where a topological space X satisfies the (mild) assumption in Lemma 4.46.

Example 4.47. Let X be a topological space, and κ a regular uncountable cardinal, e.g, $\kappa = \omega_1$.

(1) Assume that X satisfies the condition of Lemma 4.46. Then X is necessarily T_1 because each point can be written as the intersection of some of the closed subsets K_j.

(2) Assume that X is metrizable. Then X satisfies the assumption of Lemma 4.46 for $\kappa = \omega_1$: Indeed, if $Z \subseteq X$ is quasi-compact and $n \in \mathbb{N}$, then the covering of Z by closed balls of radius $1/n$ has a finite subcovering, say by closed balls $K_{1,n}, \ldots, K_{j_n,n}$, and one can consider the countable collection of the sets $K_{j,n}$.

5. Condensed abelian groups

We can now pass to condensed abelian groups. In fact, each topos admits a category of abelian group objects as we will first discuss.

5.1. **Interlude: abelian group objects in topoi.**

Let $\mathfrak{X}$ be a topos, and assume that $\mathcal{C}$ is a site such that $\mathfrak{X} = \mathrm{Sh}(\mathcal{C})$.

Lemma 5.1. *The following categories are naturally equivalent:*

(1) *Sheaves of abelian groups on $\mathcal{C}$, i.e., the full subcategory of functors $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ such that the composition $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab} \to \mathrm{Sets}$ with the forgetful functor is a sheaf of sets.*
(2) *Pairs of $(A \in \mathfrak{X}, \mu\colon A \times A \to A)$ such that for each $X \in \mathfrak{X}$ the set $A(X) := \mathrm{Hom}_{\mathfrak{X}}(X, A)$ is an abelian group for the multiplication $A(X) \times A(X) \cong \mathrm{Hom}_{\mathfrak{X}}(X, A \times A) \to A(X)$ induced by μ.*
(3) *Tuples $(A \in \mathfrak{X}, \mu\colon A \times A \to A, 0\colon * \to A, \iota\colon A \to A)$ such that the diagrams commute, which express associativity and multiplicativity for μ, unitality of 0 for μ, and that ι provides an inverse for multiplication.*
(4) *Product-preserving functors $\mathrm{Lat} \to \mathfrak{X}$, where Lat is the category of finite free abelian groups.*

Proof. We only sketch the respective functors, and leave checking the details as an exercise. Given a sheaf $\mathcal{F}$ of abelian groups on $\mathcal{C}$, the underlying sheaf of sets A is an object of $\mathfrak{X}$, and the addition in $\mathcal{F}$ defines a multiplication $A \times A \to A$ satisfying the condition in (2). The data in (2) defines the data in (3) by the Yoneda lemma. Namely, for $Y \to X$ the maps $A(X) \to A(Y)$ will automatically be morphisms of groups (because μ is compatible with the restrictions). This implies that the inversion on $A(X)$ is underlying a natural transformation of representable functors $A(-) \to A(-)$, and thus is given by a morphism $\iota\colon A \to A$ by the Yoneda lemma. Similarly, one can construct the morphism $0\colon * \to A$. As for any $C \in \mathcal{C}$ the evaluation $A \mapsto A(C)$ commutes with limits, and the diagrams in (3) only involve products, one sees that the data in (3) yields a sheaf of abelian groups $C \to A(C)$. Finally, given a functor $F\colon \mathrm{Lat} \to \mathfrak{X}$, we set $A := F(\mathbb{Z})$ with multiplication $A \times A \cong F(\mathbb{Z} \times \mathbb{Z}) \to F(\mathbb{Z})$ induced by the addition $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. Similarly, one can equip A with a unit $0\colon * \cong F(\{0\}) \to A$, and an inverse $\iota\colon A \to A$ (induced by the multiplication by -1 on $\mathbb{Z}$). Conversely, given a sheaf $\mathcal{F}$ of abelian groups on $\mathcal{C}$, we can construct a product-preserving functor $\mathrm{Lat} \to \mathfrak{X}$ by sending $L \in \mathrm{Lat}$ to the sheaf $C \mapsto \mathrm{Hom}_{\mathrm{Ab}}(L, \mathcal{F}(S))$. □

Definition 5.2. We let $\mathrm{Ab}(\mathfrak{X})$ be the category of abelian group objects in $\mathfrak{X}$, i.e., any of the equivalent categories in Lemma 5.1.

We will freely use the different viewpoints from Lemma 5.1.

Theorem 5.3. *The category $\mathrm{Ab}(\mathfrak{X})$ is a Grothendieck abelian category, that is,*

(1) *it is abelian*[21] *and has all limits/colimits,*
(2) *filtered colimits in $\mathrm{Ab}(\mathfrak{X})$ are exact,*
(3) *there exists a generator $\mathcal{G} \in \mathrm{Ab}(\mathfrak{X})$, i.e., if $A \in \mathrm{Ab}(\mathfrak{X})$ such that $\mathrm{Hom}_{\mathfrak{X}}(\mathcal{G}, A) = 0$, then $A = 0$,*
(4) *$\mathrm{Ab}(\mathfrak{X})$ has enough injectives.*[22]

Proof. We leave the first two assertions as an exercise, the existence of a generator will be discussed in Remark 5.5. By Grothendieck's theorem, the exactness of filtered colimits and the existence of a generator already imply the existence of enough injectives. However, one can also give a direct argument, [Sta17, Tag 01DL]. In fact, one can construct a functor $J\colon \mathrm{Ab}(\mathfrak{X}) \to \mathrm{Ab}(\mathfrak{X})$ and a natural transformation $\mathrm{Id} \to J$, such that for each $A \in \mathrm{Ab}(\mathfrak{X})$ the morphism $A \to J(A)$ is an injection into an injective object. □

Similarly to the case of sets one has a notion of a "free abelian group".

Lemma 5.4. *The forgetful functor $\mathrm{Ab}(\mathfrak{X}) \to \mathfrak{X}$ has a left adjoint $\mathfrak{X} \to \mathrm{Ab}(\mathfrak{X}),\ X \mapsto \mathbb{Z}[X]$.*

Proof. If $\mathfrak{X} = \mathrm{Sh}(\mathcal{C})$, then we can set $\mathbb{Z}[X]$ as the sheafification of the functor $C \mapsto \mathbb{Z}[X(C)]$ and check that it satisfies the desideratum. □

It is not clear how to describe $\mathbb{Z}[X](C)$ in general. Nevertheless, this abstract construction allows us to check the existence of a generator.

---

[21] We leave it to the reader to check the definition of an abelian category.

[22] $\mathcal{I} \in \mathrm{Ab}(\mathfrak{X})$ is injective if its contravariant Hom-functor is exact.

Remark 5.5. If $\mathfrak{X} = \mathrm{Sh}(\mathcal{C})$ and the objects in $\mathcal{C}$ form a set (which can always be assumed), then we can conclude that $\mathcal{G} := \bigoplus_{C \in \mathcal{C}} \mathbb{Z}[h_C^\sharp]$ is a generator for $\mathrm{Ab}(\mathfrak{X})$. Indeed, for $A \in \mathrm{Hom}_{\mathrm{Ab}(\mathfrak{X})}(\mathbb{Z}[h_C^\sharp], A) \cong \mathrm{Hom}_{\mathfrak{X}}(h_C^\sharp, A) \cong A(C)$ and A is zero if and only if all $A(C)$ are zero.

Remark 5.6. In general, the "free abelian sheaves" $\mathbb{Z}[X]$ are far from being projective, e.g., $\mathrm{Hom}_{\mathrm{Ab}(\mathfrak{X})}(\mathbb{Z}[*], -)$ is the global section functor for $\mathfrak{X}$, which very often is not right exact (it yields sheaf cohomology). In fact, enough projective objects exist rarely in $\mathrm{Ab}(\mathfrak{X})$: in any category with enough projective objects products are exact, but this happens rarely.

There does exist a convenient class of topoi for which *countable* products are exact.

Definition 5.7. A topos $\mathfrak{X}$ is called replete if for any sequential diagram

$$\ldots \to \mathcal{G}_2 \twoheadrightarrow \mathcal{G}_1 \twoheadrightarrow \mathcal{G}_0$$

of surjections in $\mathfrak{X}$, the projections $\varprojlim_{i \in \mathbb{N}} \mathcal{G}_i \to \mathcal{G}_j$ are surjective for each $j \in \mathbb{N}$.

Lemma 5.8. *Let $\mathfrak{X}$ be a replete topos. Then countable products are exact in* $\mathrm{Ab}(\mathfrak{X})$.

Proof. Let $\varphi_n \colon F_n \to G_n$ be a morphism of sequential diagrams $\{F_n\}_{n \in \mathbb{N}}$, $\{G_n\}_{n \in \mathbb{N}}$ such that each φ_n and each $F_{n+1} \to F_n \times_{G_n} G_{n+1}$ is surjective. Then we claim that $\varprojlim_{n \in \mathbb{N}} F_n \to \varprojlim_{n \in \mathbb{N}} G_n$ is surjective. Indeed, given a morphism $X \to \varprojlim_{n \in \mathbb{N}} G_n$, i.e., a compatible collection of morphisms $s_n \colon X \to G_n$, then we can inductively find compatible surjections $X_n \to X$ and lifts $t_n \colon X_n \to F_n$ of s_n: set $X_0 := X \times_{G_0} F_0$ and $X_{n+1} = X_n \times_{F_n \times_{G_n} G_{n+1}} F_{n+1}$. By repleteness, $\varprojlim_{n \in \mathbb{N}} X_n \to X$ is surjective, and thus $\varprojlim_{n \in \mathbb{N}} F_n \to \varprojlim_{n \in \mathbb{N}} G_n$ is surjective.

Given now a countable collection of surjections $\varphi_i \colon \mathcal{G}_i \to \mathcal{F}_i$, we need to see that their product is again surjective. But this follows from the previous claim, by considering the surjections $\prod_{i=0}^n \mathcal{G}_i \to \prod_{i=0}^n \mathcal{F}_i$ with inverse limit $\prod_{n \in \mathbb{N}} \varphi_n$. □

Example 5.9.
(1) The topos of sets is replete.
(2) Topoi associated to topological sets are often not replete as a countable limit of open coverings need not be an open cover anymore.
(3) For any cardinal κ the topos $\mathrm{CondSet}_\kappa$ is replete: each countable inverse limit of surjections of κ-profinite sets is again a surjection (using Corollary 3.24).

Frequently, we will also use the following variant for module categories in topoi.

Definition 5.10. Let $\mathfrak{X}$ be a topos, and let $\Lambda \in \mathfrak{X}$ be a ring object. We let $\mathrm{Sh}_\Lambda(\mathfrak{X})$ be the category of pairs $(M \in \mathrm{Ab}(\mathfrak{X}), \mu \colon \Lambda \times M \to M)$ such that for each $X \in \mathfrak{X}$, the map $\Lambda(X) \times M(X) \to M(X)$ makes $M(X)$ into a $\Lambda(X)$-module.

Lemma 5.11. *Let $\mathfrak{X}$ be a topos, and let $\Lambda \in \mathfrak{X}$ be a ring object. Then $\mathrm{Sh}_\Lambda(\mathfrak{X})$ is a Grothendieck abelian category, and the forgetful functor $\mathrm{Sh}_\Lambda(\mathfrak{X}) \to \mathrm{Ab}(\mathfrak{X})$ has a left adjoint $M \mapsto \Lambda[M]$.*

Proof. We leave the first part as an exercise. For the second, one can construct $\Lambda[M]$ as the sheafification of $X \in \mathfrak{X} \mapsto \Lambda(X)[M(X)]$, where the latter is the pointwise free $\Lambda(X)$-module on $M(X)$. □

One also has the following formal constructions.

Lemma 5.12. *Let $\mathfrak{X}$ be a topos, and $\Lambda \in \mathfrak{X}$ a ring object. Then $\mathrm{Sh}_\Lambda(\mathfrak{X})$ has a unique closed, symmetric monoidal structure $(-) \otimes_\Lambda (-)$, which commutes with colimits in each variable, and makes the left adjoint $\mathfrak{X} \to \mathrm{Sh}_\Lambda(\mathfrak{X})$, $X \mapsto \Lambda[X]$ symmetric monoidal.*

Here, $\Lambda[X]$ denotes the free Λ-module on X, i.e., $\Lambda[-]$ is the left adjoint to the forgetful functor $\mathrm{Sh}_\Lambda(\mathfrak{X}) \to \mathfrak{X}$ (and thus $\Lambda[X]$ should not be confused with the notation in Lemma 5.11).

Proof. One again constructs $M \otimes_\Lambda N$ as the sheafification of $C \mapsto M(C) \otimes_{\Lambda(C)} N(C)$, and checks the desired properties. □

Remark 5.13. The internal Hom $\underline{\mathrm{Hom}}_\Lambda(-,-)$ from Lemma 5.12 admits the following description: let $M, N \in \mathrm{Sh}_\Lambda(\mathfrak{X})$, and $C \in \mathfrak{X}$, then

$$\underline{\mathrm{Hom}}_\Lambda(M,N)(C) \cong \mathrm{Hom}_{\mathrm{Sh}_\Lambda(\mathfrak{X})}(\Lambda[C], \underline{\mathrm{Hom}}_\Lambda(M,N)) \cong \mathrm{Hom}_{\mathrm{Sh}_\Lambda(\mathfrak{X})}(\Lambda[C] \otimes_\Lambda M, N).$$

Remark 5.14. Let $* \in \mathfrak{X}$ be the terminal object. Then the sheafification of $C \mapsto \mathbb{Z}$ defines a sheaf $\underline{\mathbb{Z}}$ of rings in $\mathfrak{X}$, and $\mathrm{Sh}_{\underline{\mathbb{Z}}}(\mathfrak{X}) \cong \mathrm{Ab}(\mathfrak{X})$. Hence, one deduces from Lemma 5.12 the existence of a natural tensor structure and of internal Hom's in $\mathrm{Ab}(\mathfrak{X})$. We will simplify notation and write $\mathbb{Z}$ instead of $\underline{\mathbb{Z}}$ if $\mathfrak{X}$ is understood.

5.2. **Generalities on condensed abelian groups.** Let κ be an uncountable cardinal.

Definition 5.15. We let CondAb_κ be $\mathrm{Ab}(\mathrm{CondSet}_\kappa)$, i.e., the category of functors $\mathrm{Prof}_\kappa^{\mathrm{op}} \to \mathrm{Ab}$ whose composition with the forgetful functor is a κ-condensed set. We call objects in CondAb_κ condensed abelian groups.

Specializing Lemma 5.8, Theorem 5.3 to condensed sets we arrive at the following theorem.

Theorem 5.16. *The category* CondAb_κ *is Grothendieck abelian, and countable products in* CondAb_κ *are exact.*

Proof. This follows from Lemma 5.8 and Theorem 5.3. □

In the case that κ is a strong limit cardinal, one obtains even better properties.

Theorem 5.17. *Assume that κ is a strong limit cardinal. Assume that κ is a strong limit cardinal. Then* CondAb_κ *identifies with the category* $\mathrm{Fun}^\times(\mathrm{ExtDisc}_\kappa^{\mathrm{op}}, \mathrm{Ab})$ *functors* $\mathrm{ExtDisc}_\kappa^{\mathrm{op}} \to \mathrm{Ab}$ *preserving finite product. In particular,* CondAb_κ *has enough projectives, given by* $\mathbb{Z}[S]$ *for* $S \in \mathrm{ExtDisc}_\kappa$ *and all products in* CondAb_κ *are exact.*

Proof. As in Lemma 3.59 one verifies that $\mathrm{CondAb} \cong \mathrm{Fun}^\times(\mathrm{ExtDisc}_\kappa^{\mathrm{op}}, \mathrm{Ab})$. Moreover, one verifies that $\mathbb{Z}[T]$ for $T \in \mathrm{ExtDisc}_\kappa$ is projective, because $\mathcal{F} \mapsto \mathcal{F}(T) = \mathrm{Hom}_{\mathrm{CondAb}_\kappa}(\mathbb{Z}[T], \mathcal{F})$ is exact (because any surjection $T' \to T$ splits). We can conclude that a morphism $\mathcal{F} \to \mathcal{G}$ is a surjection in CondAb_κ if and only if $\mathcal{F}(T) \to \mathcal{G}(T)$ is a surjection for any $T \in \mathrm{ExtDisc}_\kappa$. This implies that all products are exact in CondAb_κ. □

In the case $\kappa = \omega_1$ Theorem 5.17 does not apply, but one has the following very strong replacement. We note that the statement is surprising as $\mathbb{Z} \cup \{\infty\}$ is not projective in Prof_{ω_1} (the surjection $2\mathbb{N} \cup \{\infty\} \coprod (2\mathbb{N}+1) \cup \{\infty\} \to \mathbb{N} \cup \{\infty\}$ does not split).[23]

Theorem 5.18. *Assume $\kappa = \omega_1$. Then* $\mathbb{Z}[\mathbb{N} \cup \{\infty\}] \in \mathrm{CondAb}_{\omega_1}$ *is internally projective, i.e., the internal Hom-functor* $\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\omega_1}}(\mathbb{Z}[\mathbb{N} \cup \infty], -) \colon \mathrm{CondAb}_{\omega_1} \to \mathrm{CondAb}_{\omega_1}$ *is exact.*

Proof. We note that $\mathbb{Z} \in \mathrm{CondAb}_{\omega_1}$ is internally projective because $\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\omega_1}}(\mathbb{Z}, M) \cong M$ for any $M \in \mathrm{CondAb}_{\omega_1}$ (because $\mathbb{Z}[S] \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}[S]$ for any $S \in \mathrm{Prof}_{\omega_1}$). We note that $\mathbb{Z}[\mathbb{N} \cup \{\infty\}] \cong P \otimes \mathbb{Z}$ for $P := \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}\infty$. Hence, it suffices to show that P is internally projective. Let $A \to B$ be a surjection of light condensed abelian groups. We need to see that for any $S \in \mathrm{Prof}_{\omega_1}$ and any map $g \colon \mathbb{Z}[S] \otimes_{\mathbb{Z}} P \to B$ there exists a covering $T \to S$ such that the map $\mathbb{Z}[T] \otimes_{\mathbb{Z}} P \to B$ lifts to A. We have

$$\mathbb{Z}[S] \otimes_{\mathbb{Z}} P \cong \mathbb{Z}[S \times (\mathbb{N} \cup \{\infty\})]/\mathbb{Z}[S \times \infty]$$

by Lemma 5.12 and right-exactness of $\mathbb{Z}[S] \otimes_{\mathbb{Z}} (-)$. Thus, the map g identifies with a map

$$g' \colon S \times (\mathbb{N} \cup \{\infty\}) \to B,$$

sending $S \times \{\infty\}$ to 0. As $A \to B$ is a surjection, there exists a covering $f \colon S' \to S \times (\mathbb{N} \cup \{\infty\})$ and a map $h \colon S' \to A$ lifting g. For $n \in \mathbb{N}$ set $S'_n := f^{-1}(S \times \{n\})$ (so that $S'_n \to S \times \{n\}$ is still surjective). By Lemma 5.19 we can find retractions $r_n \colon S' \to S'_n$ to the injections $S'_n \to S$. We can combine these retractions into a commutative diagram of disjoint unions of profinite sets

$$\begin{array}{ccc} S' \times \mathbb{N} & \xrightarrow{\coprod_n r_n} & S' \\ & \searrow^{\coprod_n f \circ r_n} & \downarrow f \\ & & S \times (\mathbb{N} \cup \{\infty\}) \end{array}$$

We can extend $S' \times \mathbb{N} \to S'$ to a map $S'' \to S'$ of light profinite sets, which is then necessarily surjective. Indeed, we can first extend to a surjection $\beta(S' \times \mathbb{N}) \to S'$, then choose a countable boolean subalgebra $A \subseteq \mathrm{Cont}(\beta(S' \times \mathbb{N}), \mathbb{F}_2)$, which contains the image of $\mathrm{Cont}(S', \mathbb{F}_2)$, and then set $S'' := \mathrm{Spec}(A)$. Moreover, we may assume that $S' \times \mathbb{N} \to S''$ is injective, with a complement $D \subseteq S''$. By Lemma 5.19 we can find a retraction $r \colon S'' \to D$. Let $h \colon S'' \to S' \to A$ be the composition. Then $h - h \circ r$ induces a map

$$g'' \colon \mathbb{Z}[S'']/\mathbb{Z}[D] = \mathbb{Z}[S'] \otimes_{\mathbb{Z}} P \to A,$$

which lifts g as desired. Indeed, we can apply Lemma 5.20 to the compactifications S'' and $S' \times (\mathbb{N} \cup \{\infty\})$ with boundaries D and $S \times \{\infty\}$, to see that $\mathbb{Z}[S'']/\mathbb{Z}[D] = \mathbb{Z}[S'] \otimes_{\mathbb{Z}} P$. Now, $h - h \circ r$ restricts to 0 on D (because r is a retraction). This implies that g'' exists. To see

[23] On the other hand for $S \in \mathrm{ExtDisc}_\kappa$ the κ-condensed abelian group $\mathbb{Z}[S]$ is not internally projective, cf. [Sch20, Proposition 3.7].

that g'' lifts g, we note that clearly $h\colon S'' \to A$ lifts g' (because already f lifts g'), and that $h \circ r\colon D \to S'' \to A \to B$ is the zero map (because $g'\colon S \times (\mathbb{N} \cup \{\infty\}) \to B$ maps $S \times \{\infty\}$ to 0 and $S'' \to S' \to S \times (\mathbb{N} \cup \{\infty\})$ maps D to $S \times \{\infty\}$: take a sequence $(s'_n, n) \in S' \times \mathbb{N}$ converging to some $d \in D$, then $r_n(s'_n, n) \in S'_n$, and this maps to $S \times \{n\} \subseteq S \times (\mathbb{N} \cup \{\infty\})$. Thus, d has to map to $S \times \{\infty\}$. □

Lemma 5.19. *Let S be a light profinite set. Then any injection $f\colon S \to T$ with $T \in \mathrm{Prof}$ admits a splitting, i.e., there exists a map $g\colon T \to S$ with $f \circ g = \mathrm{Id}_S$.*

In other words, S is an injective object in Prof.

Proof. Write $S = \varprojlim_{n\in\mathbb{N}} S_n$ as a countable cofiltered inverse limit of finite discrete sets S_n. Replacing S_n by the image of S in S_n, we may assume that $S_{n+1} \to S_n$ is surjective for any $n \in \mathbb{N}$. By induction, we can construct subsets $\Phi_n \subseteq S$ such that the composition $\Phi_n \to S \to S_n$ is bijective and $\Phi_n \subseteq \Phi_{n+1}$. Because f is injective, we can find a decomposition of T into $\sharp(\Phi_n)$-many clopen subsets, or equivalently a map $T \to T_n$ such that the composition $\Phi_n \to T \to T_n$ is bijective. Moreover, we may construct the T_n in a compatible way and get maps $T_{n+1} \to T_n$, i.e., refine the clopen decompositions. This constructs the desired splitting g as the inverse limit of the maps $T \to T_n$ as $S \to \varprojlim_{n\in\mathbb{N}} T_n$ is a continuous bijection of light profinite sets. □

Lemma 5.20. *Let U be a locally profinite set. Let $U \to S, U \to S'$ be two compactifications to light profinite sets, with boundaries $D \subseteq S$ and $D' \subseteq S'$. Then there exists a canonical isomorphism*

$$\mathbb{Z}[S]/\mathbb{Z}[D] \cong \mathbb{Z}[S']/\mathbb{Z}[D'],$$

which restricts to the identity on any clopen subset of S or S', which is contained in U.

Proof. Any compactification S of U maps to the one-point compactification of U by a map, which is the identity on U, and the one-point compactification of U is light profinite (as it admits a surjection from S or S'). Hence, we may assume that there exists a surjective map $S \to S'$ under U. This implies that S' is the pushout $D' \cup_D S$ of condensed sets. As $\mathbb{Z}[-]\colon \mathrm{CondSet}_{\omega_1} \to \mathrm{CondAb}_{\omega_1}$ is a left adjoint, this implies that $\mathbb{Z}[S'] \cong \mathbb{Z}[D'] \oplus_{\mathbb{Z}[D]} \mathbb{Z}[S]$. This in turn implies that $\mathbb{Z}[S]/\mathbb{Z}[D] \cong \mathbb{Z}[S']/\mathbb{Z}[D']$ because for any commutative diagram of exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

in an abelian category, the left square is a pushout square if and only if the map $C \to C'$ is an isomorphism. □

We note that the functor $\underline{(-)}\colon \mathrm{Top} \to \mathrm{CondSet}_\kappa$ induces a natural limit preserving functor $\underline{(-)}\colon \{$ topological abelian groups $\} \to \mathrm{CondAb}_\kappa$ (in fact any functor preserving finite products preserves abelian group objects).

Similarly, given a topological ring Λ one obtains a natural limit preserving

$$\underline{(-)}\colon \{\text{ topological } \Lambda\text{-modules }\} \to \mathrm{CondAb}_{\underline{\Lambda}} := \mathrm{Sh}_{\underline{\Lambda}}(\mathrm{CondAb}_\kappa),$$

where $\underline{\Lambda}$ is naturally a κ-condensed ring.

Remark 5.21. From Lemma 5.12 we can conclude that there exists a tensor product

$$- \otimes_{\mathbb{Z}} -\colon \mathrm{CondAb}_\kappa \times \mathrm{CondAb}_\kappa \to \mathrm{CondAb}_\kappa, (A, B) \mapsto A \otimes_{\mathbb{Z}} B$$

(and similarly $- \otimes_\Lambda -$ for Λ a condensed ring). This tensor product is however not very "topological" in nature: because sheafification does not change the value on $*$ (as any cover of $*$ splits) we get that

$$(A \otimes_{\mathbb{Z}} B)(*) = A(*) \otimes_{\mathbb{Z}} B(*)$$

is the usual algebraic tensor product over $\mathbb{Z}$.

Lemma 5.12 also supplies an internal Hom $\underline{\mathrm{Hom}}_{\mathrm{CondAb}_\kappa}(A, B)$ for $A, B \in \mathrm{CondAb}_\kappa$. This is closely related to the internal Hom in (κ-compactly generated) topological spaces. Namely, we note the following consequence of Lemma 4.7.

Lemma 5.22. *Let A, B be Hausdorff topological abelian groups, and assume that A is κ-compactly generated. Then there is a natural isomorphism of condensed abelian groups*

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}_\kappa}(\underline{A}, \underline{B}) \cong \underline{\underline{\mathrm{Hom}}_{\mathrm{Ab(Top)}}(A, B)},$$

where $\underline{\mathrm{Hom}}_{\mathrm{Ab(Top)}}(A,B)$ *is the subspace of* $\underline{\mathrm{Hom}}_{\mathrm{Top}}(A,B)$ *of group homomorphisms (and* $\underline{\mathrm{Hom}}_{\mathrm{Top}}(A,B)$ *is equipped with the compact-open topology from Definition 4.14).*

As B is Hausdorff and A κ-compactly generated, $\underline{\mathrm{Hom}}_{\mathrm{Ab(Top)}}(A,B)$ is a closed subspace.

Proof. By Lemma 4.7 we have an isomorphism

$$\Phi\colon \underline{\mathrm{Hom}}_{\mathrm{CondSet}_\kappa}(\underline{A},\underline{B}) \to \underline{\underline{\mathrm{Hom}}_{\mathrm{Top}}(A,B)},$$

and we need to see that Φ preserves the respective internal Hom's as *abelian* groups. This statement is clear after evaluating on $*$, as $\underline{\mathrm{Hom}}_{\mathrm{CondSet}_\kappa}(\underline{A},\underline{B})(*) = \mathrm{Hom}_{\mathrm{CondSet}_\kappa}(\underline{A}(*)_{\mathrm{top}},B)$ (using that A is κ-compactly generated). From here, we see that for any $S\in\mathrm{Prof}_\kappa$ the map Φ sends $\underline{\mathrm{Hom}}_{\mathrm{CondAb}_\kappa}(\underline{A},\underline{B})(S)$ to $\mathrm{Cont}(S,\underline{\mathrm{Hom}}_{\mathrm{Ab(Top)}}(A,B)) \subseteq \mathrm{Cont}(S,\underline{\mathrm{Hom}}_{\mathrm{Top}}(A,B))$ because this can be checked pointwise on S. Now assume that $f\in \mathrm{Cont}(S,\underline{\mathrm{Hom}}_{\mathrm{Ab(Top)}}(A,B))$ is given. An inverse under Φ yields a map $S\times\underline{A}\to\underline{B}$ of condensed sets, of which we have to show that it is the restriction of a map $\mathbb{Z}[S]\otimes_{\mathbb{Z}}\underline{A}\to\underline{B}$ in CondAb_κ. We have a right exact sequence

$$\mathbb{Z}[\underline{A}\times\underline{A}]\to\mathbb{Z}[\underline{A}]\to\underline{A}\to 0,$$

where the first map sends $[(a_1,a_2)]$ to $[a_1+a_2]-[a_1]-[a_2]$ (this works sectionwise on presheaves, and then can be sheafified). The map $S\times\underline{A}\to\underline{B}$ in $\mathrm{CondSet}_\kappa$ yields by adjunction a map $\mathbb{Z}[S]\times_{\mathbb{Z}}\mathbb{Z}[\underline{A}]\to\underline{B}$, and it suffices (by tensoring the above sequence with $\mathbb{Z}[S]$) to show that its composition to $\mathbb{Z}[S]\otimes_{\mathbb{Z}}\mathbb{Z}[\underline{A}\times\underline{A}]=\mathbb{Z}[S\times\underline{A}\times\underline{A}]$ vanishes. But this means equivalently that one needs to see that the map $S\times\underline{A}\times\underline{A}\to\underline{B}$ vanishes. This can be checked after evaluation on $*$, where it follows from the fact that $f(s,-)\colon A\to B$ is a group homomorphism for any $s\in S$. □

5.3. **Metrizable locally compact abelian groups.** It is time to discuss examples for condensed abelian groups, and thereby of condensed sets. From now on, we assume that $\kappa=\omega_1$, for simplicity, and we omit it from CondSet, CondAb.

We first discuss (metrizable) locally compact abelian groups.

Lemma 5.23. *The functor*

$$\Phi\colon \{\ \textit{metrizable locally compact abelian groups}\ \}\to\mathrm{CondAb},\ \ A\mapsto\underline{A}$$

is fully faithful and exact in the following sense: if $A\xrightarrow{f}B\xrightarrow{g}C$ *is a* strict *exact sequence, i.e., the quotient topology on* $\mathrm{Im}(f)=\ker(g)$ *agrees with the subspace topology, then* $\underline{A}\to\underline{B}\to\underline{C}$ *is exact.*

Proof. Fully faithfulness of Φ follows from Lemma 4.3. Exactness follows from Lemma 4.36 because quotient maps of topological groups are open. More precisely, the functor $\underline{(-)}\colon\mathrm{Top}\to\mathrm{CondAb}$ preserves equalizers, so that we can reduce to the case that $C=\{0\}, B=\ker(g)$ and the statement that $\underline{A}\to\underline{B}$ is surjective if $A\to B$ is a quotient map. □

Lemma 5.24. *Let* $f\colon A\to B$ *be a surjection of topological groups, which is a quotient map. Then* f *is open.*

Proof. Let $V\subseteq A$ be open. We need to see that $f^{-1}(f(V))$ is open. But this subset of A is the union over the open sets kV for $k\in\ker(f)$, hence open. □

Example 5.25.
(1) Examples of (metrizable) locally compact abelian groups[24] are finite dimensional $\mathbb{R}$-vector spaces, discrete abelian groups, (countable) products of $S^1=\mathbb{R}/\mathbb{Z}$, (countable) products of $\mathbb{Z}_p$'s (for varying primes p), adèles/idèles,....
(2) In general, a locally compact abelian group A is a product $A\cong\mathbb{R}^n\times A'$ with A' admitting a compact open subgroup (thus, A' is an extension of a discrete group by a compact group). A detailed analysis of the category of locally compact abelian groups is done in [HS07].
(3) If A is a locally compact abelian group, then the space $A^\vee:=\mathrm{Hom}_{\mathrm{cont}}(A,S^1)$ of continuous group homomorphisms $A\to S^1$ with the compact-open topology, the "Pontryagin dual" of A, is again a locally compact abelian group, and $A\to(A^\vee)^\vee$ is an isomorphism ("Pontryagin duality"). If A is metrizable, then $A^\vee$ need not be metrizable, e.g., if A is an uncountable direct sum of $\mathbb{Z}$'s.

The functor Φ is compatible with duality in the following sense.

Lemma 5.26. *Let* A *be a metrizable locally compact abelian group, then* $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(\underline{A},\underline{S^1})\cong\underline{A^\vee}$.

Proof. This follows from Lemma 5.22. □

[24] Locally compact abelian groups are always assumed to be Hausdorff.

Remark 5.27. In cases, where $A^\vee$ is not metrizable, one has to be careful that $\underline{A^\vee}(*)_{\mathrm{top}}$ is given by $A^\vee$ with the ω_1-compactly generated topology. Similarly, it is not clear whether for an uncountable set I and $A = \prod_I S^1$, one has $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(\underline{A}, \underline{S^1}) \cong \bigoplus_I \underline{\mathbb{Z}} = \underline{A^\vee}$, even after evaluating on $*$.

Lemma 5.28. *Let A be a separable metrizable locally compact abelian group, i.e., A contains a countable dense subset. Then $A^\vee$ is metrizable, and the biduality map*

$$\underline{A} \to \underline{\mathrm{Hom}}_{\mathrm{CondAb}}(\underline{A^\vee}, \underline{S^1})$$

is an isomorphism.

Proof. If A is separable, then the compact-open topology on $\mathrm{Hom}_{\mathrm{Ab(Top)}}(A, S^1)$ is metrizable[25] □

We will later compute the internal Ext-groups $\underline{\mathrm{Ext}}^*_{\mathrm{CondAb}}$ for (separable) metrizable locally compact abelian groups, and verify the following non-trivial facts:

(1) $\underline{\mathrm{Ext}}^*_{\mathrm{CondAb}}(\underline{\mathbb{R}}, \mathbb{Z}) = 0$,

(2) $\underline{\mathrm{Ext}}^i_{\mathrm{CondAb}}(\underline{\mathbb{R}}, \underline{\mathbb{R}}) = \begin{cases} \mathbb{R}, & i = 0, \\ 0, & i > 0 \end{cases}$.

5.4. **Banach spaces over $\mathbb{Q}_p$.** Let p be a prime. We note that $\mathbb{Q}_p$ is a topological ring, and thus $\underline{\mathbb{Q}_p} \in \mathrm{CondSet}$ is naturally a ring object. More precisely, $S \in \mathrm{Prof}_{\omega_1}$ is sent to the ring $\underline{\mathbb{Q}_p}(S) = \mathrm{Cont}(S, \mathbb{Q}_p)$ of continuous $\mathbb{Q}_p$-valued functions on S. By compactness of S, we can conclude that $\mathrm{Cont}(S, \mathbb{Q}_p) = \mathrm{Cont}(S, \mathbb{Z}_p)[1/p]$.

Lemma 5.29. *The functor $\underline{(-)}$ induces an fully faithful embedding of the category of metrizable topological $\mathbb{Q}_p$-vector spaces into* $\mathrm{CondAb}_{\mathbb{Q}_p} := \mathrm{Sh}_{\mathbb{Q}_p}(\mathrm{CondSet})$.

Proof. This follows from Lemma 4.3. □

More concrete examples of metrizable topological $\mathbb{Q}_p$-vector spaces are given by p-adic Banach spaces. We let

$$|-|\colon \mathbb{Q}_p \to \mathbb{R}$$

be the p-adic valuation sending p to $1/p$.

Definition 5.30. Let V be a $\mathbb{Q}_p$-vector space.

(1) A (non-archimedean) norm on V is a function $\|-\| : V \to \mathbb{R}$ such that
  - $\|av\| = |a|\|v\|$ for $a \in \mathbb{Q}_p$ and $v \in V$,
  - $\|v + w\| \leq \max(\|v\|, \|w\|)$ for any $v, w \in V$, and
  - $\|v\| = 0$ if and only if $v = 0$.

(2) A topological $\mathbb{Q}_p$-vector space V is a p-adic Banach space if its topology can be defined by a non-archimedean norm and V is complete, i.e., Cauchy sequences converge in V.[26]

(3) A p-adic Banach space is called separable if it contains a dense countable subset.

If $\|-\|$ defines the topology on V, then we call the pair $(V, \|-\|)$ a normed p-adic Banach space. Due to the non-archimedean triangle inequality, p-adic Banach spaces have a very simple structure.

Lemma 5.31. *Let $(V, \|-\|)$ be a normed p-adic Banach space. Then $L := \|-\|^{-1}([0,1])$ is a p-adically complete $\mathbb{Z}_p$-module, in fact, $L \cong \widehat{\bigoplus}_I \mathbb{Z}_p := (\bigoplus_I \mathbb{Z}_p)^\wedge_p$ and $V = L[1/p]$. Moreover, L is open.*

Proof. The fact that L is a $\mathbb{Z}_p$-module follows from the strong triangle inequality. The fact that the topology on L is p-adic follows from the observation that $p^n L = \|-\|^{-1}([0, 1/p^n])$ for $n \geq 1$. As V is complete the closed subset L is as well complete. We note that L is open, again by the strong triangle inequality (L is a union of cosets of the open subgroup $\|-\|^{-1}([0,1))$ of V). It remains to see that $L \cong \widehat{\bigoplus_I \mathbb{Z}_p}_p^\wedge$. Let a_i, $i \in I$, such that their images form a basis in the $\mathbb{F}_p$-vector space L/pL. We get a map $M := \bigoplus_{i \in I} \mathbb{Z}_p e_i \to L$ sending e_i to a_i. By p-completeness of L, this map extends to a map $f\colon N := M^\wedge_p \to L$, which is an isomorphism mod p (using that $M/p \cong N/p$). As

[25]Take a countable dense subset $a_1, a_2, \ldots \in A$, and a countable system $K_1, K_2, \ldots$ of compact neighborhoods of the unit of A. Considering finite unions of products $K_i a_j$, we can construct a sequence $L_1 \subseteq L_2 \subseteq \ldots$ of compact subspaces of A such that any compact subset $K \subseteq A$ is contained in some L_i, i.e., A is called hemicompact. Taking the sup-norm on L_n (with respect to some fixed metric d on S^1), we can construct a pseudo-metric d_n on$X := \mathrm{Hom}_{\mathrm{Top}}(A, S^1)$. The quantity $d(f,g) := \sum_{n=1}^\infty \frac{1}{2^n} \frac{d_n(f,g)}{1+d_n(f,g)}$ defines a metric on X, inducing the topology. Being a subspace of X, $A^\vee$ is therefore metrizable as well.

[26]In principle, one could also ask for a norm for which the non-archimedean triangle inequality is replaced by the usual triangle inequality. This leads to a more general notion, but it does not seem as relevant.

L is p-complete and p-torsionfree (being a subspace of V), this implies that f is an isomorphism: if $f(n) = 0$, then $n \equiv 0 \mod p$ (as f is an isomorphism mod p), and so $n = pn_1$ for some $n_1 \in N$. But then $f(n_1) = 0$ as L is p-torsion free, and one can show eventually that $n \in \cap_{i=0}^{\infty} p^i M = \{0\}$ (using that M is p-adically separated). The proof of surjectivity is left as an exercise. $\square$

Remark 5.32. If $V = (\widehat{\bigoplus}_I \mathbb{Z}_p)[1/p]$ is a p-adic Banach space, then it is usually no harm to assume that I is countable (equivalently V separable). Indeed, we can write I as the ω_1-filtered colimit over countable subsets $J \subseteq I$, and consequently V as the *uncompleted* filtered colimit of the $(\widehat{\bigoplus}_J \mathbb{Z}_p)[1/p]$. In fact, the colimit being ω_1-filtered implies that it commutes with the p-adic completion (which is a countable limit).

Lemma 5.33. *Let V be a p-adic Banach space. Then $\underline{V} \in \mathrm{CondAb}_{\mathbb{Q}_p}$ is a quasi-separated $\underline{\mathbb{Q}_p}$-module, and if $V = (\widehat{\bigoplus}_{i\in\mathbb{N}} \mathbb{Z}_p)[1/p]$, then $\underline{V} = (\underline{\widehat{\bigoplus}_{i\in\mathbb{N}} \mathbb{Z}_p})[1/p]$ and*

$$\underline{\widehat{\bigoplus}_{i\in\mathbb{N}} \mathbb{Z}_p} = \bigcup_{f\colon \mathbb{N}\to\mathbb{N},\ f(n)\to\infty} p^{f(n)} \prod_{n\in\mathbb{N}} \underline{\mathbb{Z}_p}.$$

Proof. That $\underline{V}$ is quasi-separated follows from the fact that V is Hausdorff (Lemma 4.21). We can note that $L := \widehat{\bigoplus_{i\in\mathbb{N}} \mathbb{Z}_p} \subseteq \prod_{i\in\mathbb{N}} \mathbb{Z}_p$ is the subspace of those sequences $(a_i)_{i\in\mathbb{N}}$ such that $|a_i| \to 0$ for $i \to \mathbb{N}$. In other words, those sequences that lie in $N_f := \prod_{i\in\mathbb{N}} p^{f(i)}\mathbb{Z}_p \subseteq \prod_{i\in\mathbb{N}} \mathbb{Z}_p$ for some function $f\colon \mathbb{N} \to \mathbb{N}$ which tends to ∞. Clearly, we have an inclusion $\underline{N_f} \to \underline{L}$. Let $N \subseteq \underline{L}$ be the (filtered) union of all these subspaces. Let $S \in \mathrm{Prof}_{\omega_1}$, and $S \to L$ a continuous map. We need to see that S factors over some continuous map $S \to N_f$. But this is easy to guarantee as for each $n \geq 0$ the quotient L/p^n is discrete, and thus the continuous map $S \to L/p^n$ has to factor over a finite, discrete quotient of S. From Lemma 4.6 and the fact that $p^n L \subseteq V$ is open (Lemma 5.31) we can deduce that $\underline{V} = \underline{\varinjlim(L \xrightarrow{p} L \xrightarrow{p} \dots)} \cong \underline{L}[1/p]$. $\square$

Example 5.34. A typical example for a p-adic Banach space is the space of continuous functions $\mathrm{Cont}(S, \mathbb{Q}_p)$ on a compact Hausdorff space with the sup-norm. In this case, the open unit ball is given by $\mathrm{Cont}(S, \mathbb{Z}_p)$.

Duality for Banach spaces is usually an important topic. From the condensed viewpoint, the duality takes however a slightly different form: the dual should not be assumed to be a Banach space again.

Lemma 5.35. *Let $V = (\widehat{\bigoplus}_I \mathbb{Z}_p)[1/p]$ be a separable p-adic Banach space. Then*

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{Q}_p}}(\underline{V}, \mathbb{Q}_p) \cong (\prod_I \underline{\mathbb{Z}_p})[1/p].$$

Proof. Set $L := \widehat{\bigoplus}_I \mathbb{Z}_p$. By Lemma 5.22, or rather a slight modification to $\mathbb{Q}_p$-vector spaces,

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{Q}_p}}(\underline{V}, \mathbb{Q}_p) \cong \underline{\mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p)}$$

identifies with the space of continuous morphisms of $\mathbb{Q}_p$-Banach spaces. By continuity each continuous morphism $f\colon V \to \mathbb{Q}_p$ will map some $p^n L$ to $\mathbb{Z}_p$. This (together with Lemma 4.6) implies that

$$\underline{\mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p)} \cong \underline{\mathrm{Hom}_{\mathbb{Z}_p}(L, \mathbb{Z}_p)}[1/p].$$

Now,

$$\underline{\mathrm{Hom}_{\mathbb{Z}_p}(L, \mathbb{Z}_p)} \cong \underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{Z}_p}}(\underline{L}, \underline{\mathbb{Z}_p}) \cong \varprojlim_{n\in\mathbb{N}} \underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{Z}_p/p^n}}(\underline{L}/p^n, \underline{\mathbb{Z}/p^n}) \cong \prod_{i\in I} \mathbb{Z}/p^n$$

using $\underline{L}/p^n \cong \underline{L/p^n} \cong \bigoplus_I Z/p^n$ (Lemma 4.24). $\square$

Definition 5.36. A (separable) p-adic Smith space M is an object of the form $(\prod_I \underline{\mathbb{Z}_p})[1/p]$ for some (countable) set I.

Theorem 5.37. *The functor $\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{Q}_p}}(-, \mathbb{Q}_p)$ defines an anti-equivalence between separable p-adic Banach spaces and separable p-adic Smith spaces.*

Proof. By Lemma 5.35 it is sufficient to show that

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{Z}_p}}((\prod_I \underline{\mathbb{Z}_p})[1/p], \mathbb{Q}_p) \cong \widehat{\bigoplus_I} \mathbb{Z}_p[1/p]$$

for a countable set I. We have

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{Z}_p}}((\prod_I \underline{\mathbb{Z}_p})[1/p], \mathbb{Q}_p) \cong \underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{Z}_p}}(\prod_I \mathbb{Z}_p, \mathbb{Q}_p) \cong \underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{Z}_p}}(\prod_I \mathbb{Z}_p, \mathbb{Z}_p)[1/p],$$

using that $\prod_I \mathbb{Z}_p$ is qcqs to commute the colimit $\mathbb{Q}_p = \varinjlim(\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{p} \ldots)$. We continue:

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{Z}_p}}(\prod_I \mathbb{Z}_p, \mathbb{Z}_p)[1/p] \cong \varprojlim_{n\in\mathbb{N}} \underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{Z}_p}}(\prod_I \mathbb{Z}_p, \mathbb{Z}/p^n) \cong \varprojlim_{n\in\mathbb{N}} \underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{Z}_p}}(\prod_I \mathbb{Z}/p^n, \mathbb{Z}/p^n) \cong \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n$$

as desired using exactness of countable products, Lemma 5.22 and Lemma 3.33. More precisely, we note that

$$\underline{\mathrm{Hom}}_{\mathrm{Ab(Top)}}(\prod_I \mathbb{Z}/p^n, \mathbb{Z}/p^n) \cong \bigoplus_I \mathbb{Z}/p^n$$

is discrete for the compact-open topology (this is always true for $\underline{\mathrm{Hom}}_{\mathrm{Top}}(K, D)$ if K is compact and D discrete as follows readily from the definition). This finishes the proof. □

To pass to a good *abelian* category for p-adic functional analysis, Clausen/Scholze considered *solid* $\mathbb{Q}_p$-modules. We will discuss this very convenient class of condensed abelian groups in Section 5.7. For now, we give the following ad-hoc, but equivalent, definition.

Definition 5.38. We let $\mathrm{Solid}_{\mathbb{Q}_p} \subseteq \mathrm{CondAb}$ be the smallest full subcategory, which contains $(\prod_I \mathbb{Z}_p)[1/p]$ for any countable set I, and which is stable under all colimits, limits and extensions in CondAb.

The important message (for now) is that in "condensed functional analysis over $\mathbb{Q}_p$" the basic building blocks are not p-adic Banach spaces, but (separable) p-adic Smith spaces. For a good reason: we will see that $(\prod_I \mathbb{Z}_p)[1/p]$ is a compact projective object in $\mathrm{Solid}_{\mathbb{Q}_p}$, cf. Section 5.7.

Heuristically, it is not unconveivable that p-adic Smith spaces are better behaved than (condensed abelian groups associated to) p-adic Banach spaces, they are a countable union of a compact subspace, while a p-adic Banach space is in general an uncountable colimit of compact subspaces (as in 5.33).

5.5. **Banach spaces over $\mathbb{R}$.** We now give a glimpse on the condensed view point on Banach spaces over $\mathbb{R}$. The situation is far more complicated than in 5.4. We let $|-|\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ be the usual absolute value.

Definition 5.39. Let V be a $\mathbb{R}$-vector space.

(1) A norm on V is a function $\|-\| : V \to \mathbb{R}$ such that
   - $\|av\| = |a|\|v\|$ for $a \in \mathbb{R}$ and $v \in V$,
   - $\|v + w\| \leq \|v\| + \|w\|$ for any $v, w \in V$, and
   - $\|v\| = 0$ if and only if $v = 0$.

(2) A topological $\mathbb{R}$-vector space V is a Banach space over $\mathbb{R}$ if its topology can be defined by a norm and V is complete, i.e., Cauchy sequences converge in V.

(3) A Banach space over $\mathbb{R}$ is called separable if it contains a dense countable subset.

Example 5.40. As in 5.34 a typical example for a Banach space over $\mathbb{R}$ is given by the Banach space $\mathrm{Cont}(K, \mathbb{R})$ of continuous functions on a compact Hausdorff space K. Again the norm is the sup-norm.

We start our discussion with the calculation of the condensed duals.

Lemma 5.41. *Let V be a Banach space over $\mathbb{R}$. Then*

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\underline{\mathbb{R}}}}(\underline{V}, \underline{\mathbb{R}}) = \bigcup_{c\in\mathbb{R}_{\geq 0}} c\underline{L}$$

for the compact Hausdorff subspace $L := \underline{\mathrm{Hom}}_{\mathbb{R},\mathrm{cont}}(V, \mathbb{R})^{\leq 1}$ of $\mathbb{R}$-linear maps $f\colon V \to \mathbb{R}$ whose operator norm $\|f\| := \sup\{ \frac{|f(v)|}{\|v\|} \mid v \neq 0\}$ is bounded by 1 (for some fixed choice of norm on V).

Proof. By (an $\mathbb{R}$-linear variant of) 5.22 we have

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\underline{\mathbb{R}}}}(\underline{V}, \underline{\mathbb{R}}) \cong \underline{\underline{\mathrm{Hom}}_{\mathbb{R},\mathrm{cont}}(V, \mathbb{R})},$$

so we need to see that $L \subseteq \underline{\mathrm{Hom}}_{\mathbb{R},\mathrm{cont}}(V, \mathbb{R})$ is a compact subspace for compact-open topology, and that $\bigcup_{c\in\mathbb{R}_{\geq 0}} cL = \underline{\mathrm{Hom}}_{\mathbb{R},\mathrm{cont}}(V, \mathbb{R})$. The last assertion follows from the fact that continuous, $\mathbb{R}$-linear maps $V \to \mathbb{R}$ are bounded in the operator norm. The compactness of L is the so-called Banach-Alaoglu theorem. In fact, the natural map $L \to \prod_{v\in V}[-1, 1],\ \phi \mapsto (\varphi(v))_{v\in V}$ is a closed embedding. □

The subspaces $\underline{\mathrm{Hom}}_{\mathbb{R},\mathrm{cont}}(V,\mathbb{R})^{\leq 1} \subseteq \underline{\mathrm{Hom}}_{\mathbb{R},\mathrm{cont}}(V,\mathbb{R})$ and $\prod_I \mathbb{Z}_p \subseteq \underline{\mathrm{Hom}}_{\mathbb{Q}_p,\mathrm{cont}}(\widehat{\bigoplus_I \mathbb{Z}_p},\mathbb{Q}_p)$ play similar roles. In particular, we arrive at the notion of a real Smith space (viewed from the condensed lense, in the classical definition of a Smith space one just considers the dual in topological spaces).

Definition 5.42. A Smith space over $\mathbb{R}$ is a condensed $\mathbb{R}$-vector space, which is isomorphic to $\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\underline{\mathbb{R}}}}(\underline{V},\underline{\mathbb{R}})$ for a Banach space V.[27]

By abuse of notation, we call the condensed $\underline{\mathbb{R}}$-vector spaces of Banach spaces again Banach spaces.

Theorem 5.43. *The functor* $\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\underline{\mathbb{R}}}}(-,\underline{\mathbb{R}})$ *induces an exact anti-equivalence between Banach and Smith spaces*

Proof. The anti-equivalence of Smith spaces and Banach spaces is classical, and in the condensed world was proven in [Sch19, Theorem 4.7.]. □

Remark 5.44. Real Banach spaces are (huge) filtered colimits of real Smith spaces, [Sch19, Corollary 3.7], similarly to 5.33.

Now, the discussion diverges heavily from the one for $\mathbb{Q}_p$ - there is no immediate analog of the notion of a solid $\mathbb{Q}_p$-vector spaces, the category generated by real Smith spaces does not have the same stability properties, [Sch19, Lecture IV]. Discussing this goes beyond the scope of this lecture (it leads to liquid $\mathbb{R}$-vector spaces, [Sch19, Theorem 6.5]). We only mention that it is related to the existence of extensions

$$0 \to V \to E \to W \to 0$$

of topological $\mathbb{R}$-vector spaces, such that V, W are Banach, but E not ([Sch19, Lecture V]).

[27]The actual condensed definition in [Sch19, Definition 4.2] is different, but equivalent by [Sch19, Theorem 4.7].

5.6. **Free condensed abelian groups.** Given a condensed set X, we recall that we have constructed the "free condensed abelian group" $\mathbb{Z}[X]$ on X in 5.4. It is characterized by a natural isomorphism

$$\mathrm{Hom}_{\mathrm{CondAb}}(\mathbb{Z}[X], A) \cong \mathrm{Hom}_{\mathrm{CondSet}}(X, A) =: A(X)$$

for $A \in \mathrm{CondAb}$. By construction, 5.21 we have

$$\mathbb{Z}[X] \otimes_{\mathbb{Z}} \mathbb{Z}[Y] \cong \mathbb{Z}[X \times Y]$$

for $X, Y \in \mathrm{CondSet}$, i.e., the functor $X \mapsto \mathbb{Z}[X]$ is (naturally) symmetric monoidal if CondSet is given its cartesian symmetric monoidal structure. In particular, if X happens to be a commutative monoid, e.g., $X = \underline{\mathbb{N}}$, then $\mathbb{Z}[X]$ is naturally a commutative ring (the "condensed monoid ring") through the multiplication

$$\mathbb{Z}[X] \otimes_{\mathbb{Z}} \mathbb{Z}[X] \cong \mathbb{Z}[X \times X] \to \mathbb{Z}[X]$$

induced by the multiplication $X \times X \to X$ (similarly the unit of $\mathbb{Z}[X]$ is the morphism $\mathbb{Z} \cong \mathbb{Z}[*] \to \mathbb{Z}[X]$ induced by the unit of X).

Remark 5.45. Condensed monoid rings can produce slightly bizarr objects like the condensed ring $\mathbb{Z}[\underline{\mathbb{R}}]$ associated to the monoid $\mathbb{R}$ under addition. Namely, for each $r \in \mathbb{R}$, one obtains the morphism $T^r \colon \mathbb{Z} \to \mathbb{Z}[\underline{R}]$ induced by the inclusion $* \to \mathbb{R}, * \mapsto r$. Thus, $\mathbb{Z}[\underline{\mathbb{R}}]$ can be thought of as the ring "$\mathbb{Z}[T^{\mathbb{R}}]$" of Laurent polynomials $\sum_{r\in\mathbb{R}} a_r T^r$, but in a way that reflects the continuity in $\mathbb{R}$.

In general, $\mathbb{Z}[X]$ for $X \in \mathrm{CondSet}$ could be anything, it is for example not clear how to describe maps $T \to \mathbb{Z}[X]$ for $T \in \mathrm{CondSet}$, though $\mathbb{Z}[X](* = \mathbb{Z}[X(*)]$ (as every cover of $*$ split). Thus the following proposition is useful.

Proposition 5.46. *Let S be a light profinite sets. Write $S = \varprojlim_{i\in\mathbb{N}} S_i$ with S_i finite discrete.*

(1) *The natural map $\mathbb{Z}[S] \to \varprojlim_{i\in\mathbb{N}} \mathbb{Z}[S_i]$ is injective, and the image is given by the condensed set*

$$\bigcup_{c>0} (\varprojlim_{i\in\mathbb{N}} \mathbb{Z}[S_i]_{\ell^1 \le c}) \subseteq \varprojlim_{i\in\mathbb{N}} \mathbb{Z}[S_i],$$

where $\mathbb{Z}[S_i]_{\ell^1\le c} = \{\sum_{s\in S_i} a_s[s_i] \mid \sum_{s\in S_i} |a_s| \le c\}$ for the absolute value $|-|$ on $\mathbb{Z}$. In particular, $\mathbb{Z}[S]$ is a quasi-separated condensed abelian group.

(2) *The natural map $\underline{\mathbb{Z}[S](*)_{\mathrm{top}}} \to \mathbb{Z}[S]$ is an isomorphism and thus $\mathbb{Z}[S]$ is the condensed abelian group associated with a topological abelian group.*

The definition of $\mathbb{Z}[S_i]_{\ell^1\le c}$ might need some explanation: note that $\mathbb{Z}[S_i]$ is the condensed set associated with the discrete abelian group $\bigoplus_{s\in S_i} \mathbb{Z}[s_i]$. In particular, $\mathbb{Z}[S_i]_{\ell^1\le c}$ is literally just (the condensed set associated with) a finite set. We note that by the triangle inequality for $|-|$, the projections $\mathbb{Z}[S_j] \to \mathbb{Z}[S_i]$ send $\mathbb{Z}[S_j]_{\ell^1\le c}$ to $\mathbb{Z}[S_i]_{\ell^1\le c}$ (thus the transition maps are well-defined).

Proof. Formally, $\mathbb{Z}[S]$ is the sheafification of the functor

$$T \in \mathrm{Prof}_{\omega_1} \mapsto \mathbb{Z}[\mathrm{Cont}(T, S)].$$

We have $\mathbb{Z}[S](*) \cong \mathbb{Z}[S(*)] =\cong \bigoplus_{s\in S} \mathbb{Z}$, and this embeds into $\varprojlim_{i\in\mathbb{N}} \mathbb{Z}[S_i](*) \cong \prod_{s\in S} \mathbb{Z}$. Now assume that $f \in \mathbb{Z}[\mathrm{Cont}(T, S)]$ maps to 0 in $\varprojlim_{i\in\mathbb{N}} \mathbb{Z}[S_i](T)$. In particular, for all $t \in T$, the specialization $f(t) \in \mathbb{Z}[S](*)$ is zero. We have to see that there is some finite covering $\{T_m \to T\}_m$ by light profinite sets $T_m \to T$ such that the image of f in each $\mathbb{Z}[\mathrm{Cont}(T_m, S)]$ is zero. We can write

$$f = \sum_{j=1}^{k} n_j [g_j]$$

for distinct continuous functions $g_j \colon T \to S$, and non-zero integers n_j. We argue by induction on k. If $k = 1$, then the claim is clear as $f(t) = g_1(t)$ was zero for any $t \in T$. Assume that $k \ge 2$, and let $1 \le j < j' \le k$. We set $T_{jj'} \subseteq T$ as the closed subset where $g_j = g_{j'}$. Then the collection of $T_{jj'}$ cover T: indeed, if $t \in T$ does not lie in any $T_{jj'}$, then all $g_j(t) \in S$ are pairwise distinct, and $\sum_{j=1}^k n_j [g_j(t)] \in \mathbb{Z}[S](*)$ is non-trivial. Thus, replacing T by $\coprod_{1\le j<j'\le k} T_{jj'}$ we may reduce to the case that $g_j = g_{j'}$ for some j, j', in which case we can reduce k and apply induction. Hence, we have checked that $\mathbb{Z}[S] \to \varprojlim_{i\in\mathbb{N}} \mathbb{Z}[S_i]$ is injective.

We now check that $\bigcup_{c>0} \underbrace{\varprojlim_{i\in\mathbb{N}} \mathbb{Z}[S_i]_{\ell^1\le c}}_{\mathbb{Z}[S]_{\ell^1\le c} :=}$ is indeed a condensed subgroup of $\varprojlim_{i\in\mathbb{N}} \mathbb{Z}[S_i]$. First of all we notice that $\mathbb{Z}[S]_{\ell^1\le c} \to \varprojlim_{i\in\mathbb{N}} \mathbb{Z}[S_i]$ is clearly injective, and hence as well their union over $c > 0$. The triangle inequality for $|-|$ implies that the componentwise additions map

$\mathbb{Z}[S]_{\ell^1\leq c}\times\mathbb{Z}[S]_{\ell^1\leq c'}$ to $\mathbb{Z}[S]_{\ell^1\leq c+c'}$. Similarly, $\bigcup_{c>0}\underbrace{\varprojlim_{i\in\mathbb{N}}\mathbb{Z}[S_i]_{\ell^1\leq c}}_{\mathbb{Z}[S]_{\ell^1\leq c}:=}$ is stable under inverses, and contains the zero element. As the map $S\to\varprojlim_{i\in\mathbb{N}}\mathbb{Z}[S_i]$ factors over $\bigcup_{c>0}\underbrace{\varprojlim_{i\in\mathbb{N}}\mathbb{Z}[S_i]_{\ell^1\leq c}}_{\mathbb{Z}[S]_{\ell^1\leq c}:=}$ (even $\mathbb{Z}[S]_{\ell^1\leq 1}$), we get a natural morphism

$$\mathbb{Z}[S]\to\bigcup_{c>0}\underbrace{\varprojlim_{i\in\mathbb{N}}\mathbb{Z}[S_i]_{\ell^1\leq c}}_{\mathbb{Z}[S]_{\ell^1\leq c}:=}$$

by the universal property of $\mathbb{Z}[S]$. It remains to see that this map is surjective. Assume that $c=n\in\mathbb{N}$. For each i we have a surjective map

$$\alpha_i\colon S_i^n\times\{1,0,-1\}^n\to\mathbb{Z}[S_i]_{\ell^1\leq n}$$

sending $((s_1,\varepsilon_1),\ldots,(s_n,\varepsilon_n))$ to $\sum_{j=1}^n\varepsilon[s_i]$. Passing to the limit, we obtain a surjection

$$\alpha\colon S^n\times\{1,0,-1\}^n\to\mathbb{Z}[S]_{\ell^1\leq n}$$

of condensed sets (associated with light profinite sets). Moreover, we have a map $\beta\colon S^n\times\{1,0,-1\}^n\to\mathbb{Z}[S]$ because we can multiply the canonical map $S^n\to\mathbb{Z}[S]^n\overset{\text{addition}}{\to}\mathbb{Z}[S]$ by any scalar in $\mathbb{Z}$. We note that the two compositions

$$S^n\times\{1,0,-1\}^n\overset{\alpha}{\to}\mathbb{Z}[S]_{\ell^1\leq n}\to\bigcup_{c>0}\underbrace{\varprojlim_{i\in\mathbb{N}}\mathbb{Z}[S_i]_{\ell^1\leq c}}_{\mathbb{Z}[S]_{\ell^1\leq c}:=}$$

and

$$S^n\times\{1,0,-1\}^n\overset{\beta}{\to}\mathbb{Z}[S]\to\bigcup_{c>0}\underbrace{\varprojlim_{i\in\mathbb{N}}\mathbb{Z}[S_i]_{\ell^1\leq c}}_{\mathbb{Z}[S]_{\ell^1\leq c}:=}$$

agree. This shows the desired surjectivity. □

5.7. **Solid abelian groups.** In this subsection we want to use 5.18 to construct the very useful class Solid $\subseteq$ CondAb of *solid abelian groups.* As an application, we will be able to show

$$\underline{Ext}^i_{\mathrm{CondAb}}(\mathbb{R}/\mathbb{Z},\mathbb{Z}) = \begin{cases} 0, & i \neq 1 \\ \mathbb{Z}, & i = 1, \end{cases}$$

with the generator in degree 1 corresponding to the extension $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$.

Definition 5.47. We set $P := \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}\infty \in \mathrm{CondAb}$.

By 5.18 the object P is internally projective in CondAb, i.e., the functor

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,-)$$

is exact. In fact, this functor commutes with *all* limits and colimits.

Remark 5.48. If A is a topological abelian group, then $\mathrm{Hom}_{\mathrm{CondAb}}(P,\underline{A})$ identifies with the set of null sequences $(a_n)_n \in A$. Thus, $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,\underline{A})$ is the space of null sequences in A.

The condensed abelian group P has the natural endomorphism

$$\mathrm{shift}\colon P \to P,$$

which is induced by the addition by 1 on $\mathbb{N} \cup \{\infty\}$ (with the convention $1 + \infty = \infty$).

Definition 5.49. A condensed abelian group $A \in \mathrm{CondAb}$ is called solid if the natural map

$$1 - \mathrm{shift}^*\colon \underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,A) \to \underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,A)$$

is an isomorphism. We denote by Solid $\subseteq$ CondAb the full subcategory of solid abelian groups.

Remark 5.50. As in 5.48 we can think of $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,A)$ as the space of null sequences in A. Then, roughly, the morphism

$$1 - \mathrm{shift}^*\colon \underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,A) \to \underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,A),\ (a_n)_n \mapsto (a_n - a_{n+1})_n$$

ought to have the inverse:

$$(b_n)_n \mapsto (\sum_{i=0}^{\infty} b_i, \sum_{i=1}^{\infty} b_i, \ldots)$$

because

$$\sum_{i=n}^{\infty} b_i - \sum_{i=n+1}^{\infty} b_i = b_n$$

and

$$\sum_{i=n}^{\infty} (a_i - a_{i+1}) = a_n.$$

Hence, being solid roughly means null sequences in A are summable.

The following implies that the category of solid abelian groups has excellent properties.

Theorem 5.51. *The category* Solid *is a Grothendieck abelian category stable under limits, colimits and extensions in* CondAb. *Furthermore:*

1. $\mathbb{Z} \in \mathrm{Solid}$,
2. *if* $M \in \mathrm{CondAb}$, $N \in \mathrm{Solid}$, *then* $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M,N) \in \mathrm{Solid}$,
3. *the inclusion* Solid $\to$ CondAb *has a left adjoint* $(-)^{\square}\colon \mathrm{CondAb} \to \mathrm{Solid}$,
4. *there is a unique symmetric monoidal structure* $\otimes^{\square}$ *on* Solid *making* $(-)^{\square}$ *symmetric monoidal,*

Proof. The stability of Solid $\subseteq$ CondAb under limits, colimits, and extensions follows immediately from the fact that the functor

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,-)$$

commutes with all limits and colimits. If $M \in \mathrm{CondAb}$, $N \in \mathrm{Solid}$, then

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M,N)) \cong \underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P \otimes M,N) \cong \underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M,\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,N)).$$

Now, the morphism $1 - \mathrm{shift}^*$ is an isomorphism on $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,N)$, so that it is also an isomorphism on $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M,N))$. Next we show that $\mathbb{Z} \in \mathrm{Solid}$ (by stability under colimits this implies that for any discrete abelian group M the condensed abelian group $\underline{M}$ is solid because it admits a presentation as a cokernel of a morphisms between direct sums of copies of $\mathbb{Z}$). We first note that by 4.7 for any $S \in \mathrm{Prof}_{\omega_1}$ we have

$$\mathrm{Hom}_{\mathrm{CondAb}}(\mathbb{Z}[S],\mathbb{Z}) \cong \mathrm{Cont}(S,\mathbb{Z}),$$

and thus $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(\mathbb{Z}[S],\mathbb{Z}) \cong \underline{\mathrm{Cont}(S,\mathbb{Z})}$, where the $\mathrm{Cont}(S,\mathbb{Z})$ is given the compact-open topology. But as S is compact, and $\mathbb{Z}$-discrete, the compact-open topology is discrete. For $\mathbb{Z}[S]$ replaced by $P = \mathbb{Z}[\mathbb{N}\cup\{\infty\}]/\mathbb{Z}\infty$, we therefore obtain that

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,\mathbb{Z}) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}e_i,\ (a_0,a_1,\ldots)\mapsto \sum_{i=0}^{\infty} a_i e_i$$

is discrete on the basis e_i (defined by evaluation on $i\in\mathbb{N}$). This implies that $1-\mathrm{shift}^*$ is an isomorphism on $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,\mathbb{Z})$ because for each null sequence $(a_n)_n\in\mathbb{Z}$, i.e., $a_i=0$ for $i\gg 0$, the sum $\sum_{i=0}^{\infty} a_i$ converges. More precisely, discreteness of $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,\mathbb{Z})$ implies that it is sufficient to check that $1-\mathrm{shift}^*$ is an isomorphism after taking sections on $*$, where the concrete argument using null sequences applies.

The existence of the left adjoint $(-)^{\square}\colon \mathrm{CondAb}\to\mathrm{Solid}$ is formal as Solid is stable under limits. To show the existence of $-\otimes^{\square}-$ on Solid, and the symmetric monoidal structure on $(-)^{\square}$ it suffices to show that if $M\to M'$ is a morphism CondAb with $M^{\square}\to M^{\square}$ an isomorphism, then $(M\otimes N)^{\square}\to(M'\otimes N)^{\square}$ is an isomorphism for any $N\in\mathrm{CondAb}$. Indeed, given this claim we get the required structures if we set

$$M\otimes^{\square} N := (M\otimes N)^{\square}$$

for any $M,N\in\mathrm{Solid}$. Now assume $M^{\square}\cong(M')^{\square}$, and let $N\in\mathrm{CondAb}$. We need to see that $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M'\otimes N,A)\to\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M\otimes N,A)$ is an isomorphism for any $A\in\mathrm{Solid}$. But

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M\otimes N,A)\cong\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M,\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(N,A))\cong\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M^{\square},\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(N,A))$$

using that $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(N,A)\in\mathrm{Solid}$. We have the same for M', which then implies the desired statement.

□

We now turn to less formal statements.

Lemma 5.52. *We have $\mathbb{R}^{\square}=0$. Moreover, if $M\in\mathrm{CondAb}_{\mathbb{R}}$, then $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M,A)=0$ for any $A\in\mathrm{Solid}$.*

Proof. It suffices to show $\mathbb{R}^{\square}=0$. Indeed, then

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M,A)=\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{R}}}(M,\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(\mathbb{R},A))\cong 0$$

because $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(\mathbb{R},A)\cong\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(\mathbb{R}^{\square},A)=0$ (using that $(\mathbb{Z}[S]\otimes\mathbb{R})^{\square}=(\mathbb{Z}[S]^{\square}\otimes\mathbb{R}^{\square})^{\square}$ for any $S\in\mathrm{Prof}_{\omega_1}$).

Intuitively, $\mathbb{R}^{\square}=0$ because in $\mathbb{R}$ not every null sequence is summable. We now provide a rigorous condensed argument. To see that $\mathbb{R}^{\square}$ is zero it suffices to show that the unit map $\mathbb{Z}\to\mathbb{R}^{\square}$ (induced by the unit map $\mathbb{Z}\to\mathbb{R},\ 1\mapsto 1$) is zero. Consider the null sequence

$$(1,1/2,1/2,1/4,1/4,1/4,1/4,\ldots)$$

in $\mathbb{R}$, and its associated map $f\colon P\to\mathbb{R}$. Let $\iota\colon\mathbb{R}\to\mathbb{R}^{\square}$ be the natural map coming from adjunction. By definition of solidification, there is a unique map $g\colon P\to\mathbb{R}^{\square}$, such that $g\circ(1-\mathrm{shift}^*)=\iota\circ f$. Consider the maps

$$F\colon\mathbb{Z}[\mathbb{N}]\to\mathbb{Z}[\mathbb{N}],\ [n]\mapsto[2n+1]+[2n+2],\ G\colon\mathbb{Z}[\mathbb{N}]\to\mathbb{Z}[\mathbb{N}],\ [n]\mapsto[2n+1].$$

These maps naturally extend to endomorphisms of P. We call the extensions again F,G. Now, we note that $(1-\mathrm{shift})\circ F\cong G\circ(1-\mathrm{shift})$ because

$$(1-\mathrm{shift})\circ F([n])=(1-\mathrm{shift})([2n+1]+[2n+2])=[2n+1]-[2n+2]+[2n+2]-[2n+3]=[2n+1]-[2n+3]$$

and

$$G\circ(1-\mathrm{shift})([n])=G([n]-[n+1])=[2n+1]-[2n+3].$$

On the other hand, $f\circ F=f$, because f represents the series $(1,1/2,1/2,\ldots)$ and $f\circ F$ the series

$$((1/2+1/2),(1/4+1/4),(1/4+1/4),\ldots)=(1,1/2,1/2,\ldots).$$

By the uniqueness of $g\colon P\to\mathbb{R}^{\square}$, we must have $g\circ G=g$. Indeed, $g\circ(1-\mathrm{shift})=\iota\circ f=\iota\circ f\circ F=\iota\circ g\circ(1-\mathrm{shift})\circ F=g\circ G\circ(1-\mathrm{shift})$. This implies that if g represents the null sequence $(x_0,x_1,\ldots)$ in $\mathbb{R}^{\square}$, then $x_n=x_{2n+1}$ for $n\in\mathbb{N}$. In particular, $x_0=x_1$. But $g\circ(1-\mathrm{shift})\colon P\to\mathbb{R}^{\square}$ lifts to $f\colon P\to\mathbb{R}$, and evaluating on $[0]\in P$ yields $x_0-x_1=\iota\circ f([0])=1$. Hence, $0=x_0-x_1=1$ in $\mathbb{R}^{\square}$ as desired. □

We can note the following consequence.

Lemma 5.53. *We have* $\underline{Ext}^i_{\mathrm{CondAb}}(\mathbb{R},\mathbb{Z})=0$ *for* $i\geq 0$*, and*

$$\underline{Ext}^i_{\mathrm{CondAb}}(\mathbb{R}/\mathbb{Z},\mathbb{Z})=\begin{cases}0, & i\neq 1\\ \mathbb{Z}, & i=1.\end{cases}$$

Here, for $M\in\mathrm{CondAb}$, the object $\underline{Ext}^i_{\mathrm{CondAb}}(M,-)\in\mathrm{CondAb}$ is the i-th right derived functor of the left exact functor $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M,-)$. We will use two facts about this construction:

(1) the cohomological δ-functor $\underline{\mathrm{Ext}}^*_{\mathrm{CondAb}}(-,-)$ maps short exact sequences in each variable to natural long exact sequences,
(2) if $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M,-)$ is exact, then

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M,\underline{\mathrm{Ext}}^i_{\mathrm{CondAb}}(N,-))\cong\underline{\mathrm{Ext}}^i_{\mathrm{CondAb}}(N,\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(M,-))$$

(this follows, e.g., from [Sta17, Tag 08J9]).

Proof. We first claim that if $M\in\mathrm{CondAb}$, and $N\in\mathrm{Solid}$, then for $i\geq 0$ also $\underline{\mathrm{Ext}}^i_{\mathrm{CondAb}}(M,N)\in\mathrm{Solid}$. Indeed, we use the above remarks, which imply

$$\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,\underline{\mathrm{Ext}}^i_{\mathrm{CondAb}}(M,N))\cong\underline{\mathrm{Ext}}^i_{\mathrm{CondAb}}(M,\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(P,N)),$$

where $1-\mathrm{shift}^*$ is an isomorphism. We note that if $M\in\mathrm{CondAb}_{\mathbb{R}}$, then $\underline{\mathrm{Ext}}^i_{\mathrm{CondAb}}(M,N)$ is naturally a condensed $\mathbb{R}$-module. Being solid if $N\in\mathrm{Solid}$, this implies that it is even a module over $\mathbb{R}^{\square}=0$, hence trivial. Thus, we have checked that if $M\in\mathrm{CondAb}_{\mathbb{R}}$ and $N\in\mathrm{Solid}$, then

$$\underline{\mathrm{Ext}}^i_{\mathrm{CondAb}}(M,N)=0,\ i\geq 0.$$

Applying this to $M=\mathbb{R}$ and $N=\mathbb{Z}$, yields the first assertion. For the second assertion, we use the short exact sequence $0\to\mathbb{Z}\to\mathbb{R}\to\mathbb{R}/\mathbb{Z}\to 0$ and that $\underline{\mathrm{Hom}}_{\mathrm{CondAb}}(\mathbb{Z},-)$ is the identity functor on CondAb (hence $\underline{\mathrm{Ext}}^i_{\mathrm{CondAb}}(\mathbb{Z},-)=0$ for $i>0$). □

We have the following simple structure of free solid abelian groups.

Theorem 5.54. *Let* $S=\varprojlim_{n\in\mathbb{N}}S_n$ *be a light profinite set, and* S_n *finite discrete. Then the natural map*

$$\mathbb{Z}[S]^{\square}\to\varprojlim_{n\in\mathbb{N}}\mathbb{Z}[S_n]$$

is an isomorphism. Furthermore, $\prod_{n\in\mathbb{N}}\mathbb{Z}$ *is a compact projective generator of* Solid.

Proof. This is proven in [RC24, Theorem 3.3.1]. □

Remark 5.55. In [RC24, Corollary 3.3.6] the natural morphism $\prod_{\mathbb{N}}\mathbb{Z}\otimes^{\square}\prod_{\mathbb{N}}\mathbb{Z}\to\prod_{\mathbb{N}\times\mathbb{N}}\mathbb{Z}$ is an isomorphism. This gives a very nice way of computing solid tensor products, [RC24, Section 3.5]. For example, we note that $\mathbb{Z}[[T]]\otimes^{\square}\mathbb{Z}[[U]]\cong\mathbb{Z}[[T,U]]$, where $\mathbb{Z}[[T]]\cong\varprojlim_n\mathbb{Z}[T]/T^n$ etc.. Using a resolution

$$0\to\mathbb{Z}[[T]]\overset{T-p}{\to}\mathbb{Z}[[T]]\to\mathbb{Z}_p\to 0,$$

one checks that

$$\mathbb{Z}_p\otimes^{\square}\mathbb{Z}_\ell=\begin{cases}\mathbb{Z}_p, & p=\ell\\ 0, & p\neq\ell.\end{cases}$$

Indeed, for the first case one uses the exactness of

$$0\to\mathbb{Z}_p[[T]]\overset{T-p}{\to}\mathbb{Z}_p[[T]]\to\mathbb{Z}_p\to 0,$$

(as can be checked on topological rings using the exactness in 5.23) and for the second that $T-\ell$ is a unit in the classical ring $\mathbb{Z}_p[[T]](*)$.

Remark 5.56. 5.18 makes it very easy to construct interesting full subcategories of CondAb. For example, call a $\mathbb{R}$-module $M\in\mathrm{CondAb}_{\mathbb{R}}$ gaseous if the map

$$1-1/2\mathrm{shift}\colon\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{R}}}(\mathbb{R}\otimes P,M)\to\underline{\mathrm{Hom}}_{\mathrm{CondAb}_{\mathbb{R}}}(\mathbb{R}\otimes P,M)$$

is an isomorphism. Then the full subcategory of gaseous $\mathbb{R}$-modules in $\mathrm{CondAb}_{\mathbb{R}}$ satisfies analogous properties to 5.51. Intuitively, a condensed $\mathbb{R}$-module M is gaseous if for any null sequence $(m_0,m_1,\ldots)$ the sum $\sum_{i=0}^{\infty}\frac{1}{2}^i m_i$ converges. From here, it is easy to verify that $\mathbb{R}$-Banach spaces are gaseous, e.g., $\mathbb{R}$ itself. It is however much more difficult to calculate the free gaseous $\mathbb{R}$-module on $S\in\mathrm{Prof}_{\omega_1}$, [CS23, Lecture 14].

A similar discussion applies to $\mathbb{R}$ replaced by $\mathbb{Q}_p$ and $1/2$ by p.

6. Cohomology

We finally discuss some examples of cohomology calculations in condensed math.

6.1. Condensed and Čech cohomology.

The first examples that we calculate are the groups

$$\mathrm{Ext}^i_{\mathrm{CondAb}}(\mathbb{Z}[X], \underline{M}),$$

where M is a discrete abelian group, and X a metrizable compact Hausdorff space. By general topos-theoretic considerations, these groups identify with the cohomology groups[28]

$$H^i_{\mathrm{cond}}(X, \underline{M})$$

of the topos of sheaves on the site $\mathrm{Prof}_{\omega_1/X}$ of light profinite sets S with a morphism to X (coverings are again given by finite, jointly surjective morphisms of light profinite sets).

Theorem 6.1. *Let X be a light compact Hausdorff space and M a discrete abelian group. Then for $i \geq 0$ there is a natural isomorphism*

$$H^i_{\mathrm{cond}}(X, \underline{M}) \cong H^i(X, M),$$

where the right hand side denote sheaf cohomology of X with values in the constant sheaf associated with M.

Proof. As $\underline{X}$ is a qcqs condensed set, the cohomology groups $H^i_{\mathrm{cond}}(X, \underline{M})$ commute with filtered colimits in M, [Sta17, Tag 0739]. Using that X is compact Hausdorff, the same holds for $H^i(X, -)$, [Ans23, Lemma 4.20]. By the long exact sequence and resolving M by free abelian groups, therefore reduces to the case that $M = \mathbb{Z}$. Let Ouv_X be the site of open subsets of X. Sending $U \in \mathrm{Ouv}_X$ to the sheaf sending $S \in \mathrm{Prof}_{\omega_1/X}$ to the set of morphisms $S \to U$ over X (which is either empty of a point), defines a morphism of topoi

$$\nu \colon \mathrm{Sh}(\mathrm{Prof}_{\omega_1/X}) \to \mathrm{Sh}(X) = \mathrm{Sh}(\mathrm{Ouv}_X).$$

It suffices to check that the morphism $\mathbb{Z} \to \nu_*(\mathbb{Z})$ is an isomorphism, and that $R^i\nu_*\mathbb{Z} = 0$ for $i > 0$ (by the Leray spectral sequence). These assertions can be checked on stalks at points in X. If $x \in X$, then the stalk of $R^i\nu_*\mathbb{Z}$ identifies (using [Sta17, Tag 09YP]) with the cohomology of $\mathbb{Z}$ on the site $\mathrm{Prof}_{\omega_1/\{x\}}$, i.e., with the cohomology of $\mathbb{Z}$ on the site Prof_{ω_1}. But every covering of $* \in \mathrm{Prof}_{\omega_1}$ splits, so that this cohomology is just given by $\mathbb{Z}$ in degree 0. □

6.2. Cohomology with coefficients in $\mathbb{R}$.

We note another calculation.

Theorem 6.2. *Let X be a metrisable compact Hausdorff space. Then $H^i_{\mathrm{cond}}(X, \mathbb{R}) = 0$ for $i > 0$ and $H^0_{\mathrm{cond}}(X, \mathbb{R}) = \mathrm{Cont}(X, \mathbb{R})$.*

Proof. The proof of [Sch, Theorem 3.3] goes through here as well. □

6.2 has the following consequence.

Theorem 6.3. *Let A be a compact metrizable abelian group. Then*

$$\underline{\mathrm{Ext}}^i_{\mathrm{CondAb}}(\underline{A}, \underline{\mathbb{R}}) = 0.$$

Proof. Using the Eilenberg-MacLane-Breen-Deligne resolution ([Sch, Theorem 4.5]), and 6.2 one checks that the statement is invariant under enlarging ω_1 to a strong limit cardinal κ. In this case, the proof of [Sch, Theorem 4.3] goes through. □

Corollary 6.4. *We have*

$$\underline{\mathrm{Ext}}^i_{\mathrm{CondAb}}(\mathbb{R}, \mathbb{R}) = \begin{cases} \mathbb{R}, & i = 0 \\ 0, & i \geq 1. \end{cases}$$

Proof. 6.3 implies that $\underline{\mathrm{Ext}}^i_{\mathrm{CondAb}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}) = 0$ for $i \geq 0$. Hence, $\underline{\mathrm{Ext}}^i_{\mathrm{CondAb}}(\mathbb{R}, \mathbb{R}) \cong \underline{\mathrm{Ext}}^i_{\mathrm{CondAb}}(\mathbb{Z}, \mathbb{R})$, which implies the result. □

---

[28] Given a topos $\mathfrak{X}$, its cohomology functors $H^i(\mathfrak{X}, -)$ are the right derived functors of the global section functor $\mathrm{Ab}(\mathfrak{X}) \to \mathrm{Ab}, \mathcal{F} \mapsto \mathrm{Hom}_{\mathfrak{X}}(*, \mathcal{F})$.

6.3. **Solid group cohomology.**

Theorem 6.5. *Let G be a light profinite group, and let M be a G-module, such that $\underline{M}$ is a solid abelian group. Then*
$$\mathrm{Ext}^i_{\mathrm{CondAb}_{\mathbb{Z}[G]}}(\mathbb{Z},\underline{M}) \cong H^i_{\mathrm{cont}}(G,M).$$

Proof. The standard resolution for abstract groups yields a resolution
$$\ldots \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$$
of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules $\mathbb{Z}[G^i]$ (this works in any topos). Moreover, the diagonal G-module $\mathbb{Z}[G^i]$ is isomorphic to $\mathbb{Z}[G]\otimes_{\mathbb{Z}}\mathbb{Z}[G^{i-1}]$ with G acting only through the left tensor factor. This implies that there exists a spectral sequence
$$\mathrm{Ext}^j_{\mathrm{CondAb}_{\mathbb{Z}[G]}}(\mathbb{Z}[G^i],M) \cong \mathrm{Ext}^j_{\mathrm{CondAb}}(\mathbb{Z}[G^{i-1}],\underline{M}) \Rightarrow \mathrm{Ext}^{i+j}_{\mathrm{CondAb}_{\mathbb{Z}[G]}}(\mathbb{Z},\underline{M}).$$
Because $\underline{M}$ is solid, we get that
$$\mathrm{Ext}^j_{\mathrm{CondAb}}(\mathbb{Z}[G^{i-1}],\underline{M}) \cong \mathrm{Ext}^j_{\mathrm{Solid}}(\mathbb{Z}_{\square}[G^{i-1}],\underline{M}) = \begin{cases}\mathrm{Cont}(G^{i-1},M), & j=0\\ 0, & j\geq 1,\end{cases}$$
because $\mathbb{Z}_{\square}[G]$ is projective in Solid by 5.54. But the continuous cohomology of M is now by definition given by the cohomology of the resulting cochain complex $\mathrm{Cont}(G^{\bullet-1},M)$. This proves the comparison. □

References

[AGV72] Michael Artin, Alexandre Grothendieck, and Jean-Louis Verdier. Théorie des topos et cohomologie étale des schémas, tome 1, 2, 3. *Séminaire de géométrie algébrique du Bois-Marie 1963–1964 (SGA 4)*, pages 1972–1973, 1972.

[Ans23] Johannes Anschütz. Lecture notes on étale cohomology. *https://janschuetz.perso.math.cnrs.fr/skripte/lecture_notes_etale_cohomology.pdf*, 2023.

[CS23] Dustin Clausen and Peter Scholze. Analytic stacks. *URl: https://people.mpim-bonn.mpg.de/scholze/AnalyticStacks.html*, 2023.

[HS07] Norbert Hoffmann and Markus Spitzweck. Homological algebra with locally compact abelian groups. *Advances in mathematics*, 212(2):504–524, 2007.

[Ked24] Kiran S. Kedlaya. Notes on condensed mathematics. *https://kskedlaya.org/papers/condensed.pdf*, 2024.

[LS] Bernard Le Stum. An introduction to condensed mathematics.

[nLa] nLab. nlab, independent family of sets. *https://ncatlab.org/nlab/show/independent+family+of+sets*.

[RC24] Juan Esteban Rodriguez Camargo. Notes on solid geometry. *https://blogs.cuit.columbia.edu/jr4460/files/2024/09/SeminarSolidGeometrynotes2-1.pdf*, 2024.

[Sch] Peter Scholze. Lectures on condensed mathematics. available at https://www.math.uni-bonn.de/people/scholze/Condensed.pdf.

[Sch19] Peter Scholze. Lectures on analytic geometry. *URl: https://www. math. uni-bonn. de/people/scholze/Analytic. pdf*, 2019.

[Sch20] Peter Scholze. Lectures on condensed mathematics and complex geometry. *URl: https://www. math. uni-bonn. de/people/scholze/Complex. pdf*, 2020.

[Sta17] The Stacks Project Authors. *Stacks Project*. http://stacks.math.columbia.edu, 2017.

[Str] Neil Strickland. The category of cghw spaces. *https://ncatlab.org/nlab/files/StricklandCGHWSpaces.pdf*.