SOLID GROUP COHOMOLOGY

Contents

1. Solid G-modules

For a topological space T we denote by

$$\underline{T}$$

the condensed set

$$S \text{ profinite } \mapsto \underline{T}(S) := \mathrm{Hom}_{\mathrm{cont}}(S, T).$$

The functor

$$\underline{(-)} \colon (\mathrm{Top}) \to \mathrm{Cond}$$

is fully faithful on compactly generated topological spaces and admits as a left adjoint the functor

$$\mathrm{Cond} \to (\mathrm{Top}),\ X \mapsto X(*)_{\mathrm{top}}$$

where

$$X(*)_{\mathrm{top}}$$

denotes the global sections of X equipped with the *compactly generated topology* (the unique topology such that $U \subseteq X(*)$ is open if and only if for any morphism $S \to X$ with S profinite the preimage of U is open under the map $S = S(*) \to X(*)$).

Let G be a locally profinite group. Then $\underline{G}$ is a condensed group.

Lemma 1.1. *Let $(\mathcal{A}, \mathcal{M})$ be an analytic associative animated ring and let $g \colon \mathcal{A} \to \mathcal{B}$ be a map of condensed animated associative rings. Then the functor*

$$\mathcal{N} : S \mapsto \mathcal{B}[S] \otimes_{\mathcal{A}} (\mathcal{A}, \mathcal{M})$$

defines an analytic animated associative ring $(\mathcal{B}, \mathcal{N})$.

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Proof. This statement can be found in [4, Proposition 12.8.]. □

Remark 1.2. Note that here the tensor product

$$-\otimes_{\mathcal{A}}(\mathcal{A},\mathcal{M})$$

is just a different notation for the derived $\mathcal{M}$-completion of the $\mathcal{A}$-module

$$\mathcal{B}[S].$$

By construction, the analytic animated associative ring $(\mathcal{B},\mathcal{N})$ depends only on the $\mathcal{M}$-completion of $\mathcal{B}$ (if $\mathcal{A}$ is commutative as then the $\mathcal{M}$-completion is symmetric monoidal).

Note that if $\mathcal{A},\mathcal{B}$ and the $\mathcal{M}[S]$ are concentrated in degree 0, the objects $\mathcal{N}[S]$ need not.

Let G be a locally profinite group, and let Λ be a (condensed or topological) ring of coefficients. *For simplicity we will assume that*

$$\Lambda\in\{\mathbb{Z},\mathbb{Z}_\ell,\mathbb{Z}/\ell^n,\mathbb{F}_\ell\}$$

for some prime ℓ, although there should exist a theory for any (commutative) analytic ring. The simplification comes mostly from the fact that the pre-analytic ring

$$(\Lambda, S\mapsto \Lambda[S]^{\blacksquare})$$

is analytic (the solidification is underived here). If $\Lambda\in\{\mathbb{Z}_\ell,\mathbb{Z}/\ell^n,\mathbb{F}_\ell\}$ the Λ is even compact, which may be useful in some arguments. We call the category of associated complete modules

$$\Lambda-\mathrm{Solid}.$$

Note that by definition a condensed Λ-module is in $\Lambda-\mathrm{Solid}$ if and only if its underlying abelian group is solid.

We denote by

$$\Lambda[\underline{G}]$$

the Λ-group ring of the condensed group $\underline{G}$.

Let us record the following consequence of Lemma 1.1.

Lemma 1.3. *The pre-analytic ring*

$$(\Lambda[\underline{G}], S\mapsto \Lambda[\underline{G}][S]^{\blacksquare})$$

(where the solidification is underived!) is analytic.

Proof. By Lemma 1.1 it suffices to see that

$$\Lambda[G][S]^{L\blacksquare}\cong\Lambda[G][S]^{\blacksquare}.$$

We can write the LHS as

$$\begin{aligned} & (\Lambda[G]\otimes_{\mathbb{Z}}\mathbb{Z}[S])^{L\blacksquare} \\ \cong\ & (\Lambda[G])^{L\blacksquare}\otimes^{L\blacksquare}_{\mathbb{Z}}\mathbb{Z}[S]^{L\blacksquare}, \end{aligned}$$

which is discrete as even both factors are projective solid abelian groups, cf. Lemma 1.4 below.[1] □

Lemma 1.4. *Let S be a locally profinite set. Then*

$$\Lambda[S]^{\blacksquare}$$

is a projective object in $\Lambda-\mathrm{Solid}$.

Proof. By writing S as a disjoint union of profinite sets (which are send by $\Lambda[-]^{\blacksquare}$) to direct sums), we can reduce to the case that S is profinite. In this case,

$$\Lambda[S]^{\blacksquare} \cong \prod_I \Lambda$$

for some set I, and thus $\Lambda[S]^{\blacksquare}$ is projective, by [5, Corollary 6.1.]. □

Note that the analytic ring

$$(\Lambda[\underline{G}], S \mapsto \Lambda[G][S]^{\blacksquare})$$

is not normalized (cf. [4, Definition 12.9.]), but its normalization is given by

$$(\Lambda[\underline{G}]^{\blacksquare}, S \mapsto \Lambda[G][S]^{\blacksquare}),$$

where $\Lambda[\underline{G}]^{\blacksquare}$ is the solidification of $\Lambda[\underline{G}]$. Following Kohlhaase, cf. [2], we write

$$\Lambda(G) := \Lambda[\underline{G}]^{\blacksquare}$$

as it is a condensed analog of the ring appearing there.

Let us now collect several possibilities to define a category of "continuous" G-Λ-modules.

(1) topological G-Λ-modules, i.e., topological Λ-modules M with a continuous action

$$G \times M \to M$$

(here Λ is given its natural topology: discrete if $\Lambda \in \{\mathbb{Z}, \mathbb{Z}/\ell^n, \mathbb{F}_\ell\}$, ℓ-adic if $\Lambda = \mathbb{Z}_\ell$).

(2) Condensed $\underline{G}$-modules, i.e., condensed Λ-modules[2] M together with an action

$$\underline{G} \times M \to M$$

in the category of condensed Λ-modules. By definition, this category is equivalent to

$$\Lambda[\underline{G}] - \mathrm{Cond},$$

i.e., to the category of modules of the condensed ring $\Lambda[\underline{G}]$ in the category of condensed abelian groups.

[1] The same argument also shows that if in Lemma 1.1 $\mathcal{A} = \mathbb{Z}$ with its solid structure, and $\mathcal{B}$ is concentrated in degree 0, then the analytic ring $(\mathcal{B}, \mathcal{N})$ is again concentrated in degree 0.

[2] More precisely, condensed $\underline{\Lambda}$-modules.

(3) The complete modules for the analytic ring

$$(\Lambda[G], S \mapsto \Lambda[G][S]^{\blacksquare}),$$

or equivalently, the complete modules for the normalized analytic ring

$$(\Lambda(G), S \mapsto \Lambda[G][S]^{\blacksquare}).$$

We denote this category by

$$G - \mathrm{Solid},$$

or $G - \mathrm{Solid}_\Lambda$, when we want to stress Λ.

(4) The category of condensed $\Lambda(G)$-modules.

From analyticity we see that

$$G - \mathrm{Solid}$$

resp.

$$D(G - \mathrm{Solid})$$

embed fully faithfully into

$$\Lambda[G] - \mathrm{Cond}, \quad \Lambda(G) - \mathrm{Cond}$$

resp.

$$D(\Lambda[G] - \mathrm{Cond}), \quad D(\Lambda(G) - \mathrm{Cond}),$$

cf. [4, Proposition 12.4.]. We don't know if

$$\Lambda(G) - \mathrm{Cond}$$

embeds fully faithfully in $\Lambda[\underline{G}] - \mathrm{Cond}$. In the following, we will mostly be interested in the category $G - \mathrm{Solid}$ (and its derived category). Note that by definition an object $M \in \Lambda[G] - \mathrm{Cond}$ is in $G - \mathrm{Solid}_\Lambda$ if and only if the underlying condensed abelian group of M is solid.

Remark 1.5. Let X be a small v-stack (i.e. a stack on the category of perfectoid spaces in characteristic p, endowed with the v-topology, satisfying a certain set-theoretic condition). In joint work in progress, Fargues-Scholze define (by v-descent from spatial diamonds) a triangulated category

$$D_{\blacksquare}(X, \Lambda)$$

of solid sheaves of Λ-modules, for $\Lambda \in \{\mathbb{Z}_\ell, \mathbb{Z}/\ell^n, \mathbb{F}_\ell\}$, with ℓ prime to p. It is a full subcategory of $D(X_v, \Lambda)$. When $X = [*/G]$, with G locally profinite, $D_{\blacksquare}(X, \Lambda)$ should exactly be $D(G - \mathrm{Solid})$.

For any topological Λ-module M the condensed set

$$\underline{M}$$

is naturally in $\Lambda[G] - \mathrm{Cond}$.

If M is discrete (or if the underlying topological abelian group is an inverse limit of discrete abelian groups), then actually

$$\underline{M} \in G - \mathrm{Solid}.$$

Let us note that the condition

$$\text{"}\underline{M}\text{ is solid"}$$

for topological abelian groups M is stable under various operations, e.g. inverse limits, ... cf. the conditions in [1, Lemma 4.3.9.]. This yields a natural source of examples of objects in $G-\mathrm{Solid}$.

2. Solid group cohomology

Let us now pass to cohomology. By definition, the cohomology in $\Lambda[G]-\mathrm{Cond}$, i.e., the derived[3] functor of

$$\mathrm{Hom}_{\Lambda[\underline{G}]}(\Lambda,-)$$

on condensed $\Lambda[\underline{G}]$-modules, computes the cohomology of the site

$$BG_{\mathrm{proet}}$$

of condensed sets with G-action from [1, Section 4.3.]. On solid coefficients, this functor can also be constructed as the derived functor of

$$\mathrm{Hom}_{G-\mathrm{Solid}}(\Lambda,-)$$

on $G-\mathrm{Solid}$, because the functor

$$D(G-\mathrm{Solid})\to D(\Lambda[G]-\mathrm{Cond})$$

is fully faithful by analyticity. For $M\in\Lambda[\underline{G}]-\mathrm{Cond}$ let us denote by

$$H^*_{\mathrm{cond}}(G,M)$$

its condensed cohomology. If M is solid we also call it the solid group cohomology of G with coefficients in M. Outside the case of solid coefficients we won't consider the derived functor of

$$\mathrm{Hom}_{\Lambda(G)}(\Lambda,-).$$

Note that

$$\Lambda\cong\mathbb{Z}\otimes_{\mathbb{Z}[\underline{G}]}\Lambda[\underline{G}]\cong\mathbb{Z}\otimes^{L\blacksquare}_{\mathbb{Z}[\underline{G}]^\blacksquare}\Lambda(G)$$

which implies that the cohomology of a solid G-module does not depend on our choice of coefficients, and we are free with taking $\Lambda=\mathbb{Z}$.

Let M be a topological G-module. We recall that the continuous cohomology $H^*_{\mathrm{cont}}(G,M)$ of M is defined as the cohomology of the complex

$$C^\bullet_{\mathrm{cont}}(G,M):\quad M\to\mathrm{Hom}_{\mathrm{cont}}(G,M)\to\mathrm{Hom}_{\mathrm{cont}}(G\times G,M)\to\dots$$

of continuous cochains. We want to relate this to the solid group cohomology of G(cf. [1, Lemma 4.3.9.] for a similar discussion).

Let H be any group in any topos $\mathcal{X}$. Then we have an exact "standard complex"

$$\ldots\to\mathbb{Z}[H\times H\times H]\to\mathbb{Z}[H\times H]\to\mathbb{Z}[H]$$

[3] The derived functor can be constructed using a projective resolution of the $\Lambda[G]$-module Λ.

(by sheafifying the usual standard complex), which is a resolution of the trivial H-module $\mathbb{Z}$. Moreover, the individual terms $\mathbb{Z}[H^i]$ for $i \geq 1$ (with the diagonal action) are free $\mathbb{Z}[H]$-modules, i.e.,

$$\mathbb{Z}[H^i] \cong \mathbb{Z}[H] \otimes_{\mathbb{Z}} \mathbb{Z}[H^{i-1}]$$

with H-action only on the left factor. In particular, we obtain for every H-module N a spectral sequence

$$E_1^{ij} = H^j_{\mathcal{X}}(H^i, N) \Rightarrow H^{i+j}(\mathcal{X}/H, N),$$

where $H^\bullet_{\mathcal{X}}(U, -)$ denotes the cohomology in the topos $\mathcal{X}$ of some object $U \in \mathcal{X}$, and

$$H^\bullet(\mathcal{X}/H, -)$$

the cohomology of the topos of H-objects in $\mathcal{X}$. Assume that

$$H^j_{\mathcal{X}}(H^i, N) = 0$$

for all $j > 0$ and $i \geq 0$. Then the above spectral sequence collapses and this shows that

$$H^\bullet(\mathcal{X}/H, N)$$

can be computed via the "standard complex with cochains in N"

$$N(*) \to N(H) \to N(H \times H) \to \ldots.$$

Let us apply this reasoning in the case that $\mathcal{X} = \mathrm{Cond}$, $H = \underline{G}$ for G locally profinite, and $N = \underline{M}$ for some topological G-module M. Then we obtain the following comparison of "condensed/solid" group cohomology with continuous group cohomology.

Lemma 2.1. *With the notation from above assume that $N = \underline{M}$ is solid. Then*

$$H^*_{\mathrm{cont}}(G, M) \cong H^*_{\mathrm{cond}}(\underline{G}, \underline{M}),$$

i.e., continuous group cohomology agrees with solid group cohomology.

Proof. As

$$\underline{M}(\underline{G}^i) = \mathrm{Hom}_{\mathrm{cont}}(G^i, M)$$

for all $i \geq 0$ the above discussion implies that it suffices to see that

$$H^j(\underline{G}^i, \underline{M}) = 0$$

for all $j > 0$. This is implied by Lemma 2.2 below. □

Lemma 2.2. *Let S be a locally profinite set and let M be a solid abelian group. Then*

$$H^j(S, M) = 0$$

for $j > 0$.

Proof. This follows from Lemma 1.4 and fully faithfulness of

$$D(\mathrm{Solid}) \subseteq D(\mathrm{Ab}(\mathrm{Cond}))$$

because

$$H^j(S, M) = \mathrm{Ext}^j_{\mathrm{Ab}(\mathrm{Cond})}(\mathbb{Z}[S], M) \cong \mathrm{Ext}^j_{\mathrm{Solid}}(\mathbb{Z}[S]^{\blacksquare}, M) = 0$$

for $j > 0$ by projectivity of $\mathbb{Z}[S]^{\blacksquare}$ in Solid. □

We denote by

$$\mathrm{Rep}^{\infty}_{\Lambda} G$$

the category of smooth representations of G on Λ-modules, i.e., Λ-modules M endowed with the discrete topology, with a continuous action

$$G \times M \to M.$$

Note that in the case $\Lambda = \mathbb{Z}_\ell$ the Λ-action $\Lambda \times M \to M$ is *not* required to be continuous for the ℓ-adic topology. On discrete topological abelian groups the functor

$$M \mapsto \underline{M}$$

is exact, and thus extends to a functor on the derived categories. As an application of the comparison of continuous and solid group cohomology we can prove the following strengthening in the case of discrete coefficients.

Proposition 2.3. *Assume that Λ is discrete, i.e., $\Lambda \in \{\mathbb{Z}, \mathbb{Z}/\ell^n, \mathbb{F}_\ell\}$. Then the functor*

$$D^+(\mathrm{Rep}^{\infty}_{\Lambda} G) \to D^+(G - \mathrm{Solid}_{\Lambda}), M \mapsto \underline{M}$$

is fully faithful and its essential image is given by all objects $C \in D^+(G -$ $\mathrm{Solid}_\Lambda)$ whose cohomology is discrete as a condensed A-module

Proof. Fix $N \in D^+(\mathrm{Rep}^{\infty}_{A} G)$, and consider the full subcategory

$$\mathcal{C} \subseteq D^+(\mathrm{Rep}^{\infty}_{\Lambda} G)$$

of objects $M \in D^+(\mathrm{Rep}^{\infty}_{\Lambda} G)$ such that the canonical morphism

$$R\mathrm{Hom}_{D^+(\mathrm{Rep}^{\infty}_{\Lambda} G)}(M, N) \to R\mathrm{Hom}_{D^+(G-\mathrm{Solid}_{\Lambda}\mathrm{Rep}^{\infty}_{\Lambda} G)}(\underline{M}, \underline{N})$$

is an isomorphism. Clearly, $\mathcal{C}$ is stable under homotopy colimits, in particular filtered colimits and geometric realizations. Thus, we may first reduce to the case that M is concentrated in degree 0 and then that $M \cong \mathrm{cInd}^G_U \Lambda$ is the compact induction of the trivial U-module Λ for some compact-open subgroup $U \subseteq G$ (as modules of these form resolve any smooth representation). But

$$\underline{\mathrm{cInd}^G_U \Lambda} \cong \Lambda(G) \otimes^{L\blacksquare}_{\Lambda(U)} \Lambda$$

and thus

$$R\mathrm{Hom}_{D^+(G-\mathrm{Solid}_{\Lambda}}(\underline{\mathrm{cInd}^G_U \Lambda}, \underline{N}) \cong R\mathrm{Hom}_{D^+(U-\mathrm{Solid}_{\Lambda}}(\Lambda, \underline{N}).$$

This reduces the claim to showing that if G profinite and $N \in D^+(\mathrm{Rep}^{\infty}_{\Lambda} G)$, then

$$R\mathrm{Hom}_{D^+(\mathrm{Rep}^{\infty}_{\Lambda} G)}(\Lambda, N) \to R\mathrm{Hom}_{D^+(G-\mathrm{Solid}_{\Lambda})}(\underline{\Lambda}, \underline{N})$$

is an isomorphism. The full subcategory of such N is triangulated and contains each object, which is concentrated in degree 0 by Lemma 2.1. We have to show that

$$\mathrm{Ext}^i_{D^+(\mathrm{Rep}^\infty_\Lambda G)}(\Lambda, N) \to \mathrm{Ext}^i_{D^+(G-\mathrm{Solid}_\Lambda)}(\underline{\Lambda}, \underline{N})$$

is an isomorphism for each $i \in \mathbb{Z}$. But for a fixed i we can reduce to the case that N is bounded by taking a canonical truncation (as we assumed $N \in D^+$). □

Remark 2.4. When $\Lambda = \mathbb{Z}/\ell^n$, $n \geq 1$, Fargues-Scholze generalize Proposition 2.3 as follows: if X is a small v-stack, one has a fully faithful embedding

$$D_{\text{ét}}(X, \Lambda) \subset D_{\blacksquare}(X, \Lambda).$$

In Proposition 2.3 the case where $\Lambda = \mathbb{Z}_\ell$ is more complicated and leads to the definition of D_{lis}.

Up to now we only considered the derived functor of the functor

$$G - \mathrm{Solid}_\Lambda \to \mathrm{Ab},\ M \mapsto \mathrm{Hom}_{\Lambda(G)}(\Lambda, M).$$

However, it is reasonable to consider as well the condensed version

$$G - \mathrm{Solid}_\Lambda \to \mathrm{Solid},\ M \mapsto \underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, M),$$

where $\underline{\mathrm{Hom}}_{\Lambda(G)}$ refers to the internal Hom in condensed abelian groups (which is automatically solid here). Of course, taking global sections (which is exact) of $R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, M)$ recovers $R\mathrm{Hom}_{\Lambda(G)}(\Lambda, M)$. Consider now a topological G-module M such that $\underline{M}$ is solid. Then $R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, \underline{M})$ can be calculated via the condensed standard complex

$$\underline{M} \to \underline{\mathrm{Hom}}(\underline{G}, \underline{M}) \to \ldots.$$

Lemma 2.5. *If G is profinite, and M a discrete G-module, then*

$$R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, \underline{M}) \cong \underline{R\Gamma(G, M)}.$$

Proof. This follows by calculating the LHS via the condensed standard complex as our assumptions imply that each

$$\underline{\mathrm{Hom}}(\underline{G}^i, \underline{M})$$

is discrete. □

Remark 2.6. The statement in Lemma 2.5 cannot be generalized to arbitrary locally profinite, or even discrete, groups G. For example, if $G := \bigoplus_{\mathbb{N}} \mathbb{Z}$ is an infinite direct sum of copies of $\mathbb{Z}$ and M is a discrete G-module with trivial action, then

$$\underline{\mathrm{Hom}}(\underline{G}, \underline{M}) \cong \underline{\prod_{\mathbb{N}} M}$$

for the product topology, while the RHS in Lemma 2.5 would be $\prod_{\mathbb{N}} M$ with the discrete topology (as we did not dare to put a topology on the continuous resp. usual cohomology groups).

Remark 2.7. The question of considering a condensed structure on cohomology, i.e., to consider

$$R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, -),$$

seems relevant in establishing (or reproving) some version of the Hochschild-Serre spectral sequence in continuous group cohomology. Namely, if $N \subseteq G$ is a closed normal subgroup then for formal reasons there exists a spectral sequence

$$E_2^{ij} = H^i_{\mathrm{cond}}(G/N, \underline{\mathrm{Ext}^j_{\Lambda(N)}}(\Lambda, M)) \Rightarrow H^{i+j}_{\mathrm{cond}}(G, M)$$

for each $M \in G - \mathrm{Solid}$. Now, one can ask the question when all terms admit an interpretation in terms of continuous group cohomology.

Remark 2.8. Let $f : X \to Y$ be a map of small v-stacks. Fargues-Scholze prove that the functor

$$Rf_{v*} : D(X_v, \Lambda) \to D(Y_v, \Lambda)$$

preserves the solid categories, and therefore induces a functor

$$Rf_* : D_{\blacksquare}(X, \Lambda) \to D_{\blacksquare}(Y, \Lambda),$$

which is a right adjoint to f^*. The special case where $f : [*/G] \to *$ is our functor

$$D(G - \mathrm{Solid}_\Lambda) \to D(\Lambda - \mathrm{Solid}),\ M \mapsto R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, M).$$

3. Finiteness conditions

Let G be a profinite group, and $\Lambda \in \{\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Z}/\ell^n, \mathbb{F}_\ell\}$. Let us start with a general result.

Lemma 3.1. *Let G be a profinite group. Then the object $\Lambda \in G - \mathrm{Solid}$ is pseudo-coherent, and thus for each $i \in \mathbb{Z}$ the functor*

$$H^i_{\mathrm{cond}}(\underline{G}, -) \colon G - \mathrm{Solid} \to \mathrm{Ab}$$

commutes with filtered colimits.

Proof. Consider the standard resolution

$$... \to \Lambda[\underline{G} \times \underline{G}] \to \Lambda[\underline{G}] \to \Lambda$$

to the trivial $\underline{G}$-module Λ and its solidification

$$... \to \Lambda[\underline{G} \times \underline{G}]^{\blacksquare} \to \Lambda[\underline{G}]^{\blacksquare} \to \Lambda$$

which is a resolution of Λ (as $\Lambda^{L\blacksquare} \cong \Lambda$). Now the claim follows because each

$$\Lambda[\underline{G^i}]$$

for $i \geq 1$ is a compact projective object in $G-\mathrm{Solid}$ as it is the base change of the compact projective solid Λ-module

$$\Lambda[\underline{G}^{i-1}]$$

(here we used that G is profinite). □

In this section we want to give sufficient conditions which guarantee that Λ is even *perfect*, at least if $\Lambda = \mathbb{F}_\ell$.

We will need the following proposition on inverse limits of compact abelian groups, which we learned from Scholze.

Proposition 3.2. *Let $A_i, i \in I$, be a cofiltered inverse system of compact abelian groups. Then $R^j \varprojlim_{i\in I} \underline{A_i} = 0$ for $j > 0$.*

Proof. Set $B_i := \mathrm{Hom}_{\mathrm{cont}}(A_i, \mathbb{R}/\mathbb{Z})$ be the Pontryagin dual of A_i. Then

$$R\varprojlim_{i\in I} \underline{A_i} = R\underline{\mathrm{Hom}}(\underline{B}, \mathbb{R}/\mathbb{Z}),$$

where

$$B := \varinjlim_{i\in I^{\mathrm{op}}} B_i$$

is the filtered colimit of the discrete groups B_i (note that $\underline{B}$ is still the filtered colimit of the $\underline{B_i}$ as each B_i is discrete). Let S be extremally disconnected. We have to show that

$$R\mathrm{Hom}(\underline{B}_{|S}, \mathbb{R}/\mathbb{Z}_{|S}) = 0$$

is concentrated in degree 0, where $|S$ denotes restriction to the slice topos Cond/S of condensed sets over S. Let

$$\nu\colon \widetilde{\mathrm{Cond}/S} \to \widetilde{S}$$

be the natural morphism to the topos of the topological spaces S, i.e., $\nu^{-1}U$ of any open set S is sent to the condensed set $\underline{U}$ over $S = \underline{S}$. Then

$$\underline{B}_{|S} = \nu^{-1}(B)$$

and thus

$$R\mathrm{Hom}(\underline{B}_{|S}, \mathbb{R}/\mathbb{Z}_{|S}) \cong R\mathrm{Hom}_{\widetilde{S}}(B, R\nu_*(\mathbb{R}/\mathbb{Z})).$$

We claim that $R\nu_*(\mathbb{R}/\mathbb{Z})$ is isomorphic to the sheaf $\underline{\mathbb{R}/\mathbb{Z}}$ sending $U \subseteq S$ open to $\mathrm{Hom}_{\mathrm{cont}}(U, \mathbb{R}/\mathbb{Z})$. Indeed, as the $U \subseteq S$ quasi-compact open form a basis for the topology it suffices to show that

$$H^*(\nu^{-1}U, \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(U, \mathbb{R}/\mathbb{Z}).$$

But this follows from [5, Theorem 3.2.] resp. [5, Theorem 3.3.]. By Lemma 3.3 $\underline{\mathbb{R}/\mathbb{Z}}$ is an injective sheaf of abelian groups on $\widetilde{S}$. Hence,

$$\mathrm{Ext}^i_{\widetilde{S}}(B, \underline{\mathbb{R}/\mathbb{Z}}) = 0$$

for $i > 0$, which finishes the proof. □

Lemma 3.3. *Let S be an extremally disconnected space. Then the abelian sheaf $\underline{\mathbb{R}/\mathbb{Z}}$ on S is injective.*

Proof. It suffices to prove (cf. Lemma 2.13 by Spaltenstein in Borel, "Intersection cohomology") that

$$\underline{\mathbb{R}/\mathbb{Z}}(U)$$

is divisible for each open subset $U \subset S$ (which follows from the vanishing of the cohomology of $\mathbb{Z}/n, n \geq 1$, on locally profinite sets), and that the restriction

$$\underline{\mathbb{R}/\mathbb{Z}}(U) \to \underline{\mathbb{R}/\mathbb{Z}}(V)$$

is a split surjection for any open subsets $V \subseteq U$ of S. By Lemma 3.4 for any open $U \subset S$ the closure $\overline{U} \subseteq S$ agrees with the Stone-Čech compactification of U. Moreover, by the condition of being extremally disconnected the closure $\overline{U}$ is again open in S. As $\mathbb{R}/\mathbb{Z}$ is compact Hausdorff we obtain that

$$\mathrm{Hom}_{\mathrm{cont}}(U, \mathbb{R}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{cont}}(\overline{U}, \mathbb{R}/\mathbb{Z}).$$

This implies the second statement as each continuous function $\overline{V} \to \mathbb{R}/\mathbb{Z}$ can be extend by zero to a continuous function $\overline{U} \to \mathbb{R}/\mathbb{Z}$. □

We learned the following observation from Scholze.

Lemma 3.4. *Let S be an extremally disconnected space and let $U \subseteq S$ be open. Then the canonical morphism $\beta U \to \overline{U}$ is an isomorphism.*

Here βU is the Stone-Čech compactification of U.

Proof. As S is compact Hausdorff $\overline{U}$ is compact Hausdorff, too. In particular, the morphism $U \to \overline{U}$ extends to $\beta U \to \overline{U}$ by the universal property of the Stone-Čech compactification. Then

$$U \times_{\overline{U}} \beta U \cong U.$$

The closure $\overline{U} \subseteq S$ is again open because S is extremally disconnected. Hence, the morphism

$$\beta U \sqcup S \setminus \overline{U}$$

is a cover of S, which is therefore split. This yields a morphism $\overline{U} \to \beta U$, which is necessarily an isomorphism over U. As the morphism $\overline{U} \to \beta U$ has closed image containing U we can see that it is surjective. But the morphism $\overline{U} \to \beta U \to \overline{U}$ is the identity and hence $\beta U \cong \overline{U}$, as desired. □

Remark 3.5. Here is a simpler proof of Proposition 3.2, proposed by Juan Esteban Rodriguez Camargo. In the following, we will use the fact that underlining a strict exact sequence of locally compact abelian groups gives a short exact sequence of condensed abelian groups. One resolves B (which is discrete):

$$0 \to \oplus_I \mathbb{Z} \to \oplus_J \mathbb{Z} \to B \to 0.$$

For any profinite set S and any index set I,

$$R\underline{\mathrm{Hom}}(\oplus_I\underline{\mathbb{Z}},\underline{\mathbb{R}/\mathbb{Z}})(S) = R\mathrm{Hom}(\mathbb{Z}[\underline{S}],\prod_I R\underline{\mathrm{Hom}}(\underline{\mathbb{Z}},\underline{\mathbb{R}/\mathbb{Z}})) = \prod_I R\mathrm{Hom}(\mathbb{Z}[\underline{S}]\otimes_{\underline{\mathbb{Z}}}\underline{\mathbb{Z}},\underline{\mathbb{R}/\mathbb{Z}})$$

$$= \prod_I R\Gamma(\underline{S},\underline{\mathbb{R}/\mathbb{Z}}) = \prod_I \underline{\mathbb{R}/\mathbb{Z}}[0],$$

by [5, Theorem 3.2.] and [5, Theorem 3.3.]. Therefore, $R\underline{\mathrm{Hom}}(\underline{B},\underline{\mathbb{R}/\mathbb{Z}})(S)$ is computed by the complex

$$\prod_J \underline{\mathbb{R}/\mathbb{Z}} \to \prod_I \underline{\mathbb{R}/\mathbb{Z}}$$

which is what one gets by underlining the dual the above resolution of B; thus it has cohomology only in degree 0.

Remark 3.6. Let X be a spatial diamond. Fargues-Scholze prove that for any cofiltered system of constructible étale sheaves $\mathcal{F}_i$, killed by some non-zero integer, and any $j > 0$,

$$R^j \varprojlim_i \mathcal{F}_i = 0$$

(where the inverse limit is taken in the category of proétale sheaves over X). This generalizes (in the finite case) Proposition 3.2, which is the special case, where X is a geometric point.

Let us note the following more concrete description of $G - \mathrm{Solid}_{\mathbb{F}_\ell}$ for G profinite.

Lemma 3.7. *Let $\mathcal{C}$ be the category of finite dimensional $\mathbb{F}_\ell$-vector spaces with a continuous G-action. The canonical functor*

$$\mathrm{IndPro}(\mathcal{C}) \to G - \mathrm{Solid}_{\mathbb{F}_\ell}$$

is an equivalence.

Proof. By Proposition 3.2 we can deduce that the functor

$$\mathrm{Pro}(\mathcal{C}) \to G - \mathrm{Solid}_{\mathbb{F}_\ell}$$

is exact. It is moreover seen to be fully faithful (by definition of the inverse limit topology). As the image consists of compact objects one can deduce the statement on ind-objects. Combining these two facts, we deduce that the image of $\mathrm{IndPro}(\mathcal{C})$ in $G-\mathrm{Solid}_{\mathbb{F}_\ell}$ is stable by kernels and cokernels. Let $M \in G - \mathrm{Solid}_{\mathbb{F}_\ell}$. One can find a surjection from a direct sum of $\mathbb{F}_\ell[G][S]^{\blacksquare}$, with S profinite, which are in $\mathrm{IndPro}(\mathcal{C})$ by construction. The kernel is again in $G - \mathrm{Solid}_{\mathbb{F}_\ell}$, so receives itself a surjective map from a direct sum of $\mathbb{F}_\ell[G][S]^{\blacksquare}$. This way, we have written M as the cokernel of a morphism between two objects in $\mathrm{IndPro}(\mathcal{C})$ and so M is itself in $\mathrm{IndPro}(\mathcal{C})$ by the above. This proves essential surjectivity. □

The same argument works with $\mathbb{F}_\ell$ replaced by $\mathbb{Z}_\ell$ or $\mathbb{Z}/\ell^n$.

Remark 3.8. Similarly, using Remark 3.6, Fargues-Scholze prove that if X is a spatial diamond, $D_{\blacksquare}(X, \mathbb{Z}_\ell)$ is the derived category of the abelian category

$$\mathrm{IndPro}(\mathcal{C}),$$

where $\mathcal{C}$ is the category of constructible étale sheaves killed by a power of ℓ. This allows them to check many properties of $D_{\blacksquare}(X, \mathbb{Z}_\ell)$ by descent to spatial diamonds and reduction to the case of constructible étale sheaves killed by a power of ℓ, previously studied by Scholze, [6].

Proposition 3.9. *Assume that G is of ℓ-cohomological dimension $\leq n$ and that $H^*(G, M)$ is finite for each finite, discrete G-module M of ℓ-power order. Then $\mathbb{F}_\ell \in G - \mathrm{Solid}_{\mathbb{F}_\ell}$ is perfect with perfect amplitude $\leq n$.*

Here by perfect we mean quasi-isomorphic to a bounded complex of compact projective objects in $G - \mathrm{Solid}_{\mathbb{F}_\ell}$.

Proof. Let $\mathcal{C} \subseteq G - \mathrm{Solid}_{\mathbb{F}_\ell}$ be the subcategory of all M for which

$$R\underline{\mathrm{Hom}}_{\mathbb{F}_\ell[G]^{\blacksquare}}(\mathbb{F}_\ell, M) \in D^{[0,n]}.$$

By assumption and the comparison Lemma 2.5, this is known for M discrete. By Lemma 3.1, $\mathcal{C}$ is stable under direct sums. By Proposition 3.2 (applied twice) and the imposed finiteness for finite coefficients, the category $\mathcal{C}$ contains all inverse of finite discrete G-modules. In particular, all $\mathbb{F}_\ell[G]^{\blacksquare}$-modules whose underlying condensed set is compact, thus especially $\mathbb{F}_\ell[G]^{\blacksquare}$, lies in $\mathcal{C}$. Moreover, cokernels of morphisms between compact $\mathbb{F}_\ell[G]^{\blacksquare}$-modules lie in $\mathcal{C}$. All of this together implies that $\mathcal{C} = G - \mathrm{Solid}_{\mathbb{F}_\ell}$, by arguing similarly as in the proof of Lemma 3.7.

Let

$$0 \to Q \to P_{n-1} \to \ldots \to P_0 \to \mathbb{F}_\ell \to 0$$

be a resolution with the P_{i-1} compact projective (for example the beginning of the standard resolution) and Q admitting a surjection from a compact projective $\mathbb{F}_\ell[G]^{\blacksquare}$-module. Then

$$\mathrm{Ext}^i_{\mathbb{F}_\ell[G]^{\blacksquare}}(Q, M) \cong H^{i+n}(G, M) = 0$$

for all $i > 0$ and all $M \in G - \mathrm{Solid}$. This implies that Q is projective, and thus that Q is compact projective. This finishes the proof. □

Remark 3.10. A similar argument applies to $\Lambda = \mathbb{Z}/\ell^n$ or $\Lambda = \mathbb{Z}_\ell$.

4. Duality

Let G be a locally profinite group. A new feature of the solid G-modules is the existence of solid group *homology*. Namely, if $M \in G - \mathrm{Solid}_\Lambda$, then the homology of

$$\Lambda \otimes^{L\blacksquare}_{\Lambda(G)} M$$

is the condensed (or solid) group homology

$$H_*^{\mathrm{cond}}(G, M).$$

It is related to solid cohomology by "trivial duality", by which we mean the following assertion, which is an immediate consequence of adjunction.

Proposition 4.1. *Let G be a locally profinite group. Then for any $M \in D(G - \mathrm{Solid}_\Lambda)$ and any $Q \in D(\mathrm{Solid}_\Lambda)$ there is a natural isomorphism*

$$R\underline{\mathrm{Hom}}_\Lambda(M \otimes^L_{\Lambda[G]^\blacksquare} \Lambda, Q) \cong R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, R\underline{\mathrm{Hom}}_\Lambda(M, Q)).$$

In particular, the dual of homology is cohomology of the dual.

Remark 4.2. Let $f : X \to Y$ be a map of small v-stacks. Fargues-Scholze prove that the functor $f^* : D_\blacksquare(Y, \Lambda) \to D_\blacksquare(X, \Lambda)$ admits a left adjoint

$$f_\natural : D_\blacksquare(X, \Lambda) \to D_\blacksquare(Y, \Lambda).$$

It is defined as follows: since f is a slice in the site, it tautologically admits a left adjoint $f_{v\natural}$; then one sets $f_\natural$ to be the solidification of $f_{v\natural}$. It satisfies the projection formula and base change. In the special case $f : [*/G] \to *$, $f_\natural$ coincides with

$$D(G - \mathrm{Solid}_\Lambda) \to D(\Lambda - \mathrm{Solid}), M \mapsto \Lambda \otimes^{L\blacksquare}_{\Lambda(G)} M.$$

Let G be a profinite group, and fix a prime ℓ. From now on we assume that G is of ℓ-cohomological dimension $n \geq 0$. Consider

$$R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, \Lambda(G)).$$

As $\Lambda(G)$ is a G-*bi*module, we see that

$$R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, \Lambda(G)) \in D(G - \mathrm{Solid}_\Lambda).$$

Following [3, II.5] we introduce for $i \in \mathbb{Z}$ the functor

$$D_i(M) := \varinjlim_{U \subseteq G} \mathrm{Hom}(H^i(U, M), \mathbb{Q}/\mathbb{Z})$$

for each discrete G-module M, where the colimit is taken over all compact-open subgroups in G (and the transition maps are the dual of the corestriction maps).

We recall that G is a dualizing group of dimension $n \in \mathbb{N}$ at ℓ if $D_i(\mathbb{Z}/p) = 0$ for $i \neq n$, cf. [3, (3.4.6)]. Define the dualizing module

$$D_\ell := \varinjlim_m D_n(\mathbb{Z}/\ell^m).$$

Then G is called a Poincaré group (at ℓ, of dimension n) if it is a dualizing group and $D_\ell \cong \mathbb{Q}_\ell/\mathbb{Z}_\ell$. We can give the following rephrasement of this condition.

Lemma 4.3. *Let G be as before a profinite group, ℓ a prime, assume that $n := \mathrm{cd}_\ell(G) < \infty$ and that $H^*(G, M)$ is finite for every finite, discrete ℓ^∞-torsion G-module M. Then G is a Poincaré group if and only if*

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}_\ell(G)}(\mathbb{Z}_\ell, \mathbb{Z}_\ell(G)) \cong \mathbb{Z}_\ell[-n]$$

if and only if

$$R\underline{\mathrm{Hom}}_{\mathbb{F}_\ell(G)}(\mathbb{F}_\ell, \mathbb{F}_\ell(G)) \cong \mathbb{F}_\ell[-n].$$

Proof. and not Considering the short exact sequences

$$0 \to \mathbb{Z}/\ell^m(G) \to \mathbb{Z}/\ell^{m+1}(G) \to \mathbb{F}_\ell(G) \to 0$$

we see that the last two conditions are equivalent to

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}/\ell^m(G)}(\mathbb{Z}/\ell^m, \mathbb{Z}/\ell^m(G)) \cong \mathbb{Z}/\ell^m[-n]$$

for all $m \geq 0$. By definition

$$\mathbb{Z}/\ell^m(G) = \mathbb{Z}/\ell^m[G]^{\blacksquare} = \varprojlim_U \mathbb{Z}/\ell^m[G/U],$$

where U runs through the compact-open subgroups of G. By Proposition 3.2 the limit is derived. Using the imposed finiteness one can conclude that the homology groups of

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}}(R\underline{\mathrm{Hom}}_{\mathbb{Z}/\ell^m(G)}(\mathbb{Z}/\ell^m, \mathbb{Z}/\ell^m(G)), \mathbb{R}/\mathbb{Z})$$

are exactly the $D_i(\mathbb{Z}/\ell^m)$. As the functor $R\underline{\mathrm{Hom}}_{\mathbb{Z}}(-, \mathbb{R}/\mathbb{Z})$ induces a duality on finite, discrete $\mathbb{Z}/\ell^m$-modules we can conclude. □

Remark 4.4. Let G be a profinite group. Let $f : X = [*/G] \to Y = *$. Pretend that functors $f_!, f^!$ are defined. Since f is proper, one has $f_! = Rf_*$. Therefore,

$$R\underline{\mathrm{Hom}}_{D_{\blacksquare}(Y,\Lambda)}(Rf_*\Lambda[G], \Lambda) = Rf_*R\underline{\mathrm{Hom}}_{D_{\blacksquare}(X,\Lambda)}(\Lambda[G], f^!\Lambda) = Rf_*f^!\Lambda.$$

The LHS can be rewritten as

$$R\underline{\mathrm{Hom}}_{\Lambda}(R\underline{\mathrm{Hom}}_{\Lambda[G]^{\blacksquare}}(\Lambda, \Lambda[G]^{\blacksquare}), \Lambda).$$

In fact using that $\Lambda[G]^{\blacksquare}$ is a G-bimodule, this object is naturally a $\Lambda[G]^{\blacksquare}$-module, which should be $f^!\Lambda$. So G being a Poincaré group is somehow saying that the dualizing complex $f^!\Lambda$ is isomorphic to a shift of Λ, which is in some sense saying that $[*/G]$ is "Λ-cohomologically smooth".

Assume from now on that G is a Poincaré group (at ℓ, of dimension n), such that $H^*(G, M)$ is finite for each finite discrete ℓ^∞-torsion G-module M. Assume that

$$\Lambda \in \{\mathbb{Z}_\ell, \mathbb{Z}/\ell^m, \mathbb{F}_\ell\}.$$

Then G acts on

$$\Lambda[-n] \cong R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, \Lambda(G)).$$

via a character

$$\chi\colon G \to \Lambda^*.$$

By $-(\chi)$ we mean in the following the twist of the G-action by χ. Let us fix an isomorphism

$$\tau\colon \Lambda(\chi)[-n] \cong R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, \Lambda(G)).$$

Then we obtain the natural transformation

$$\begin{aligned} \eta_\tau\colon \Lambda[-n] \otimes^{L\blacksquare}_{\Lambda(G)} M \\ \xrightarrow{\cong} R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, \Lambda(G))(\chi^{-1}) \otimes^{L\blacksquare}_{\Lambda(G)} M \\ \to R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, M(\chi^{-1})) \end{aligned}$$

for any $M \in D(G - \mathrm{Solid}_\Lambda)$. Here the second arrow is a special case of the more general natural transformation

$$R\underline{\mathrm{Hom}}_{\Lambda(G)}(N, T) \otimes^{L\blacksquare}_{\Lambda(G)} M \to R\underline{\mathrm{Hom}}_{\Lambda(G)}(N, T \otimes^{L\blacksquare}_{\Lambda(G)} M)$$

for $N, M \in D(G - \mathrm{Solid}_\Lambda)$ and T a $(\Lambda(G), \Lambda(G))$-bimodule.

The following theorem can be seen as a duality theorem, although it is formulated as an isomorphism of homology and cohomology (up to a shift/twist). The duality theorem [3, (3.4.6.)] can be derived from it (under our more restrictive assumptions) by combining it with Proposition 4.1.

Theorem 4.5. *Under the above assumptions, for any $M \in D(G - \mathrm{Solid}_\Lambda)$ the morphism*

$$\eta_\tau \colon \Lambda[-n] \otimes^{L\blacksquare}_{\Lambda(G)} M \to R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, M(\chi^{-1}))$$

is an isomorphism.

Proof. By Proposition 3.9, Λ is a perfect $\Lambda(G)$-module, i.e., quasi-isomorphic to a bounded complex of retracts of finite direct sums of products $\prod_I \Lambda(G)$. This implies that the functor

$$M \mapsto R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, M(\chi^{-1}))$$

commutes with arbitrary colimits. As the category $G - \mathrm{Solid}_\Lambda$ is generated by the objects $\prod_I \Lambda(G)$ for sets I (and the LHS commutes with colimits in M), we can assume that $M \cong \prod_I \Lambda(G)$ for some set I. Note that $\Lambda(G) \cong \prod_J \Lambda$ as Λ-modules and thus

$$M \cong \prod_{I \times J} \Lambda$$

as Λ-modules. We claim that

$$N \otimes^{L\blacksquare}_{\Lambda(G)} M \cong \prod_I N$$

for any compact projective object in $G - \mathrm{Solid}_\Lambda$. Passing to retracts and finite sums we may assume that

$$N \cong \prod_J \Lambda(G)$$

for some set J. Using $\Lambda(G) \cong \prod_K \Lambda$ for some set K we can rewrite this as

$$N \cong \prod_J \Lambda \otimes^{L\blacksquare}_{\Lambda} \Lambda(G)$$

by [5, Proposition 6.3.] (which holds for our particular choice of Λ, too). Therefore

$$N \otimes^{L\blacksquare}_{\Lambda(G)} M \cong \prod_J \Lambda \otimes^{L\blacksquare}_{\Lambda} M \cong \prod_{J \times I} \Lambda(G),$$

again by [5, Proposition 6.3.]. As the target of η_τ commutes with products in M. We can therefore assume $M \cong \Lambda(G)$. But then it is clear that η_τ is an isomorphism as

$$R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, \Lambda(G)) \cong \Lambda(\chi)[-n]$$

by our assumption. □

Remark 4.6. Let us highlight the crucial points in the comparison of homology and cohomology.

1) Λ is a perfect $\Lambda(G)$-module,
2) there exists an isomorphism (of solid Λ-modules)

$$\tau\colon \Lambda[-n] \cong R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, \Lambda(G))$$

for some $n \geq 0$.

Example 4.7. Here are two interesting class of examples of groups to which Theorem 4.5 applies.

- Profinite groups with an open pro-p-group H, for $p \neq \ell$ (satisfying the finitness conditions). For such a G, $\Lambda \in G - \mathrm{Solid}_\Lambda$ is compact projective and after choice of a non-trivial Haar measure

$$R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, \Lambda(G)) \cong \underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, \Lambda(G))[0] \cong \underline{\mathrm{Hom}}_{\Lambda(G)}(C(G, \Lambda), \Lambda)$$

  is the space of Λ-valued Haar measures on G. Indeed,

$$R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, \Lambda(G)) \cong R\underline{\mathrm{Hom}}_{\Lambda(H)}(\Lambda, \Lambda(H),$$

  which implies easily the above isomorphisms.
- Compact p-adic Lie groups of dimension n. Any such group G is a Poincaré group of dimension n by the work of Lazard, and the character χ is the dual of the determinant of its adjoint representation on its Lie algebra. From the work of Lazard, one can deduce, for sufficiently small G, the existence of a resolution (in condensed G-modules)

$$0 \to M_n \to M_{n-1} \to \ldots \to M_1 \to \mathbb{Z}_\ell$$

  of $\mathbb{Z}_\ell$ with $M_i \cong \mathbb{Z}_\ell[G]^{\blacksquare \binom{n}{i}}$.

Let us end this text with some questions:

(1) Assume G is *locally* profinite. Which condition on G assure that $\mathbb{F}_\ell$ is perfect?
(2) Can one recover the full [3, (3.4.6.)], and thus cover general dualizing groups (not just Poincaré groups)?
(3) Can the same strategy be applied to *locally* profinite groups?

Regarding the first point, note that perfectness implies finite ℓ-cohomological dimension, and thus for many pairs of primes ℓ, p the $\mathrm{GL}_n(\mathbb{Q}_p)$-module $\mathbb{F}_\ell$ cannot be perfect.

For the last two points, the same strategy as above works if $G = \mathbb{Z}$. Here we can even take $\Lambda = \mathbb{Z}$. Perfectness of Λ follows from the resolution

$$0 \to \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z} \to 0$$

of discrete $\mathbb{Z}[\mathbb{Z}]$-modules. We moreover obtain that

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z}, \mathbb{Z}[\mathbb{Z}]) \cong \mathbb{Z}[-1]$$

(as G-modules).[4] This is enough to apply the above strategy.

Let again $\Lambda \in \{\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Z}/\ell^m, \mathbb{F}_\ell\}$. Recall that for any profinite set S we have the equality

$$\Lambda[S]^{\blacksquare} \cong \underline{\mathrm{Hom}}_{\mathbb{Z}}(C(S, \mathbb{Z}), \Lambda).$$

This suggests that there are *two* replacements for the $(\Lambda(G), \Lambda(G))$-bimodule $\Lambda(G) = \Lambda[G]^{\blacksquare}$ in Theorem 4.5 if G is a general locally profinite group. Namely,

$$\Lambda(G) := \Lambda[G]^{\blacksquare} \cong \underline{\mathrm{Hom}}_{\mathbb{Z}}(C(S, \mathbb{Z}), \Lambda),$$

or

$$\Delta(G) := \underline{\mathrm{Hom}}_{\mathbb{Z}}(C_c(S, \mathbb{Z}), \Lambda)$$

(which also appears in [2]), where the subscript $(-)_c$ denotes functions with compact support. Let G be an ℓ-adic Lie group of dimension n, and $U \subseteq G$ a compact-open subgroup, which is a Poincaré group. Then

$$R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, \Delta(G)) \cong R\underline{\mathrm{Hom}}_{\Lambda(U)}(\Lambda, \Lambda(U)) \cong \mathbb{Z}_\ell[-n],$$

cf. [2, Proposition 3.2.], as

$$\Delta(G) \cong R\underline{\mathrm{Hom}}_{\Lambda(U)}(\Lambda(G), \Lambda(U)).$$

But it is unclear how to compute (except if $G = \mathbb{Z}$)

$$R\underline{\mathrm{Hom}}_{\Lambda(G)}(\Lambda, \Lambda(G))$$

as $\Lambda(G)$ is not coinduced from a compact-open subgroup.

References

[1] Bhargav Bhatt and Peter Scholze. The pro-étale topology for schemes. *arXiv preprint arXiv:1309.1198*, 2013.

[2] Jan Kohlhaase. Smooth duality in natural characteristic. *Advances in Mathematics*, 317:1–49, 2017.

[3] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323. Springer Science & Business Media, 2013.

[4] Peter Scholze. Lectures on analytic geometry.

[5] Peter Scholze. Lectures on condensed mathematics. available at `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`.

[6] Peter Scholze. Étale cohomology of diamonds. *ArXiv e-prints*, September 2017.

[4] The critical point is that

$$R\underline{\mathrm{Hom}}_{\underline{\mathbb{Z}[\mathbb{Z}]}}(\underline{\mathbb{Z}[\mathbb{Z}]}, \underline{\mathbb{Z}[\mathbb{Z}]}) \cong \underline{\mathbb{Z}[\mathbb{Z}]}.$$