LECTURE NOTES FOR AN INTRODUCTORY COURSE ON THE LANGLANDS PROGRAM

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Contents

1. Remarks on the lecture

These are (slightly revised) notes for an introductory course on the Langlands program.[1]

The eventual aim of the course was to present (mostly without any indication of proof) statements/objects/conjectures... occuring in the Langlands program for a general reductive group over $\mathbb{Q}$, and explain how they specialize to the case of GL_2. Another aim was to present the construction of Galois representations associated to modular forms (with an emphasis on the case of weight 1 and 2).

Originally it was planned to include more material on the local Langlands program. But this was postponed as for the winter term 2020/2021 a lecture on this at the university Bonn was announced by Peter Scholze.

To make the notes more readable I included some statements, which in the lecture I said orally.

Disclaimer: I am by far not a person with serious knowledge/understanding of the Langlands program, thus in the notes I may oversimplify/overcomplicate things, be inaccurate, or even wrong, and miss subtelties. Thus use these notes at your own risk and consult the mentioned references for definite/correct statements. Any comments/hints on the notes are welcome!

I want to thank the participants of the course for the interested questions during the lectures. Moreover, I want to thank Ben Heuer heartily for following my suggestion to give three lectures (which ones is indicated in the respective title).

[1]As the lecture was planned to run parallel to the eventually cancelled trimester program of the same name, the original title was "The arithmetic of the Langlands program". But this title was not accurate in describing what was presented in the lecture. Therefore I changed the title of these revised notes.

2. A general introduction to the Langlands program

In this lecture we want to give a rough introduction to the Langlands program, and present a first example of its arithmetic significance.

Let us recall that a connected, smooth, affine, algebraic group G over a field k is reductive if the base change $G_{\overline{k}}$ to an algebraic closure of k contains no closed normal subgroup isomorphic to the additive group $\mathbb{G}_a$, or equivalently, the unipotent radical of $G_{\overline{k}}$ is trivial.

For each prime number p we let

$$\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n, \quad \mathbb{Q}_p := \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Q}$$

be the p-adic integers resp. the field of p-adic numbers.

Let

$$\mathbb{A}_f := \{(x_p)_p \in \prod_p \mathbb{Q}_p \mid x_p \in \mathbb{Z}_p \text{ for all but finitely many primes } p\ \}$$

be the ring of finite adéles of $\mathbb{Q}$, and let

$$\mathbb{A} := \mathbb{A}_f \times \mathbb{R}$$

be the full ring of adéles of $\mathbb{Q}$.

Let us recall that $\mathbb{A}$ is a locally compact topological ring. Namely, on $\mathbb{A}_f$ there exists a unique ring topology such that the subring

$$\prod_p \mathbb{Z}_p \subseteq \mathbb{A}_f$$

is open. On $\mathbb{A}$ one can take then the product topology with the usual topology on $\mathbb{R}$. An important property is that the subspace topology on the subring $\mathbb{Q} \subseteq \mathbb{A}$, i.e., $\mathbb{Q}$ is embedded diagonally, is discrete.

The starting point for the Langlands program is a a reductive group G over $\mathbb{Q}$, the most important example being $G = \mathrm{GL}_n$ for some $n \geq 1$.

The above facts imply that $G(\mathbb{A}) \cong G(\mathbb{A}_f) \times G(\mathbb{R})$ is naturally a locally compact topological group, with $G(\mathbb{Q}) \subseteq G(\mathbb{A})$ a discrete subgroup. Namely, choose an embedding $G \subseteq \mathrm{Spec}(\mathbb{Q}[X_1, \ldots, X_m])$ for some $m \geq 1$, and take the subspace topology of induced embedding $G(\mathbb{A}) \subseteq \mathbb{A}^m$. This topology is independent of the choice of the embedding. For more details see [GH19, Theorem 2.2.1.].

We will use the following facts:

- On any locally compact topological group H there exists a right-invariant Haar measure, cf. [GH19, Section 3.2.], which is unique to a scalar in $\mathbb{R}_{>0}$.
- On $G(\mathbb{A})$ the right-invariant Haar measure is also left-invariant, and it descends to a $G(\mathbb{A})$-invariant measure on $G(\mathbb{Q})\backslash G(\mathbb{A})$, cf. [GH19, Lemma 3.5.4.], [GH19, Lemma 3.5.3].

We now introduce one object of ultimate interest in the Langlands program for G.

Definition 2.1. *We set*

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$$

as the space of measurable functions $f \colon G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ *such that*

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} |f|^2 < \infty$$

where the integration is w.r.t. the $G(\mathbb{A})$-invariant measure on $G(\mathbb{Q})\backslash G(\mathbb{A})$.

As usual, two measurable functions in this L^2-space have to be identified if they agree outside a set of measure zero.

The space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is naturally a Hilbert space via the inner product

$$(f_1, f_2) \mapsto \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} f_1(g)\overline{f}_2(g)dg.$$

The action of $G(\mathbb{A})$ on $G(\mathbb{Q})\backslash G(\mathbb{A})$ via right translation induces a (left) action of $G(\mathbb{A})$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. By right-invariance of the chosen measure on $G(\mathbb{Q})\backslash G(\mathbb{A})$ this action is unitary, i.e., preserves the inner product.

We can now state a principal aim of the Langlands program (in a very crude form):

> Decompose the Hilbert space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ as a representation of $G(\mathbb{A})$, according to arithmetic data.

We will clarify a bit what we mean by "decomposing". Set

$$\widehat{G(\mathbb{A})}$$

as the isomorphism classes of irreducible, unitary $G(\mathbb{A})$-representations (on Hilbert spaces). Because of analytic issues it is too naive to hope that there exists a direct sum decomposition

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) \text{ “}\cong\text{”} \bigoplus_{[\pi]\in\widehat{G(\mathbb{A})}} \pi^{\oplus m_\pi}$$

with $m_\pi \in \mathbb{N} \cup \{\infty\}$ multiplicity of some irreducible, unitary $G(\mathbb{A})$-representation π.

Let us discuss the example $G = \mathbb{G}_m = \mathrm{GL}_1$. In this case,

$$G(\mathbb{Q})\backslash G(\mathbb{A}) \cong \mathbb{Q}^\times\backslash\mathbb{A}^\times$$

is a locally compact abelian group, and thus we can use (abstract) Fourier theory to analyze the $\mathrm{GL}_1(\mathbb{A}) = \mathbb{A}^\times$-representaiton

$$L^2(\mathbb{Q}^\times\backslash\mathbb{A}^\times).$$

Let us recally some result on abstract Fourier theory, following [DE14, Chapter 3].

- Let A be any locally compact abelian group (like $\mathbb{R}, S^1 \cong \mathbb{R}/\mathbb{Z}, \mathbb{A}^\times, \mathbb{Z}/n, \mathbb{Q}_p, ...$).
- Let $\widehat{A}$ be the set of isomorphism classes of irreducible, unitary representations of A.
- Then $\widehat{A}$ is in bijection to the set $\mathrm{Hom}_{\mathrm{cts}}(A, S^1)$ of unitary, continuous characters $A \to S^1$, i.e., each $\pi \in \widehat{A}$ is one-dimensional, and in particular again a locally compact abelian group (via pointwise multiplication and the compact-open topology).
- Moreover, $A \cong \hat{\hat{A}}$ as topological groups via $a \mapsto (\chi \mapsto \chi(a))$ and A is discrete if and only if $\widehat{A}$ is compact.[2]

[2] "⇒" Choose $\bigoplus_I \mathbb{Z} \twoheadrightarrow A$, then $\widehat{A}$ embeds into $\prod_I S^1$. "⇐" Use $A \cong \hat{\hat{A}}$ and the definition of the compact-open topology.

The Plancherel theorem is a major result of abstract Fourier theory. For us its relevance lies in the fact that it offers a different description of the space $L^2(A)$ as an A-representation.

Theorem 2.2 (Plancherel theorem, cf. [DE14, Chapter 3.4])**.** *The Fourier transform*

$$\mathcal{F}\colon L^1(A) \cap L^2(A) \to L^2(\widehat{A})$$

defined by

$$f \mapsto (\chi \mapsto \int\limits_A f(x)\chi(x)d\mu(x))$$

extends to a unitary isomorphism

$$\mathcal{F}\colon L^2(A) \cong L^2(\widehat{A})$$

of Hilbert spaces.

The Fourier transform $\mathcal{F}$ is an A-equivariant isomorphism, when we let A act on $L^2(\widehat{A})$ via

$$a \cdot g(\chi) := \chi(a)g(\chi)$$

for $a \in A$, $g \in L^2(\widehat{A}), \chi \in \widehat{A}$. Thus,

$$L^2(A) \cong L^2(\widehat{A})$$

as A-representations.

Let us discuss the two examples $A = S^1$, and $A = \mathbb{R}$..

- If $A = S^1$, then
$$\mathbb{Z} \cong \widehat{A},\ n \mapsto \chi_n \text{ with } \chi_n(z) = z^n,\ z \in S^1.$$
Thus
$$L^2(S^1) \cong \widehat{\bigoplus_{n\in\mathbb{Z}}} \mathbb{C}\chi_n$$
is the Hilbert space direct sum of the S^1-equivariant subspaces
$$\mathbb{C}\chi_n \subseteq L^2(S^1).$$
Concretely: each L^2-function $f\colon S^1 \to \mathbb{C}$ can uniquely be written as
$$f = \sum_{n\in\mathbb{Z}} a_n\chi_n, \quad \text{where } a_n \in \mathbb{C},\ \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty.$$
Thus, we found a nice decomposition of the unitary S^1-representation $L^2(S^1)$ into irreducible subspaces. The analytic issues are not immense. Instead of a(n algebraic) direct sum of representations, we had to consider the Hilbert space direct sum $\widehat{\bigoplus_{n\in\mathbb{Z}}} \mathbb{C}\chi_n$.
- If $A = \mathbb{R}$, then $\widehat{\mathbb{R}} \cong \mathbb{R}$ via
$$\mathbb{R} \to \widehat{\mathbb{R}},\ x \mapsto \chi_x, \quad \text{where } \chi_x\colon \mathbb{R} \to S^1,\ y \mapsto e^{2\pi i xy}.$$
The isomorphism
$$\mathcal{F}\colon L^2(\mathbb{R}) \cong L^2(\mathbb{R})$$
implies therefore that each $f \in L^2(\mathbb{R})$ can be written as
$$f(x) = \int\limits_{\mathbb{R}} g(y)\chi_y(x)dy$$

for a unique function $g \in L^2(\mathbb{R})$. Thus,

$$L^2(\mathbb{R}) \cong \int_{\mathbb{R}} \chi_y dy$$

is a Hilbert *integral* of representation (cf. [GH19, Section 3.7]), a notion which is not as immediate as the Hilbert space direct sum. Note that we cannot do better: $\chi_x \notin L^2(\mathbb{R})$ for any $x \in \mathbb{R}$, and $L^2(\mathbb{R})$ contains no irreducible subrepresentation of $\mathbb{R}$, cf. [GH19, Section 3.7].[3]

Let us go back and consider $G = \mathbb{G}_m$, i.e., $A = \mathbb{A}^\times$.

- Factoring each $n \in \mathbb{Q}^\times$ into $n = \pm p_1 \dots p_k$ with p_i prime implies
$$\mathbb{Q}^\times \backslash \mathbb{A}^\times \cong \prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0}.$$
- Let us ignore the factor $\mathbb{R}_{>0}$ for the moment and look at
$$L^2(\prod_p \mathbb{Z}_p^\times) \cong \widehat{\bigoplus_\chi} \mathbb{C}\chi.$$
with $\chi\colon \prod\limits_p \mathbb{Z}_p^\times \to S^1$ all continuous characters of $\prod\limits_p \mathbb{Z}_p^\times$. Note that each character $\chi\colon \prod\limits_p \mathbb{Z}_p^\times \cong \varprojlim\limits_m (\mathbb{Z}/m)^\times \to S^1$ factors over some quotient $(\mathbb{Z}/m)^\times$.
- The characters χ are "arithmetic" data, namely
$$\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \cong \prod_p \mathbb{Z}_p^\times,$$
where $\mathbb{Q}(\mu_\infty) = \bigcup\limits_{n \in \mathbb{N}} \mathbb{Q}(e^{\frac{2\pi i}{n}})$ is the cyclotomic extension of $\mathbb{Q}$. Define
$$\alpha\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \cong \prod_{p \text{ prime}} \mathbb{Z}_p^\times$$
as the natural projection.
- Thus for each $\chi\colon \prod\limits_p \mathbb{Z}_p^\times \to S^1$ the composition $\chi \circ \alpha$ is a (continuous) 1-dimensional Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
- By the theorem of Kronecker-Weber *each* 1-dimensional Galois representation
$$\sigma\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^\times$$
is of the form $\sigma = \chi \circ \alpha$ for some χ.

Theorem 2.3 (Kronecker-Weber)**.** *The field $\mathbb{Q}(\mu_\infty)$ is the maximal abelian extension of $\mathbb{Q}$, i.e., each finite Galois extension $F/\mathbb{Q}$ with abelian Galois group $\mathrm{Gal}(F/\mathbb{Q})$ is contained in $\mathbb{Q}(\mu_\infty)$.*

Thus, we obtain the following decomposition for $G = \mathbb{G}_m$:

$$L^2(\mathbb{R}_{>0}\mathbb{Q}^\times \backslash \mathbb{A}^\times) \cong \widehat{\bigoplus_{\sigma\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^\times}} \mathbb{C}\chi_\sigma$$

[3] Also no irreducible quotient, the statement about quotients in [GH19, Lemma 3.7.1] is wrong, I think.

with χ_σ characterised by $\chi_\sigma \circ \alpha = \sigma$. This (easy) example is prototypical for what one aims for in the Langlands program. Note that modding out $\mathbb{R}_{>0}$ is convenient and rather harmless. The full space

$$L^2(\mathbb{Q}^\times \backslash \mathbb{A}^\times$$

is a Hilbert integral of the representations $L^2(\mathbb{R}_{>0}\mathbb{Q}^\times \backslash \mathbb{A}^\times)$ over the space $\widehat{\mathbb{R}} \cong \mathbb{R}$ of unitary characters of $\mathbb{R}_{>0}$.

Let us pass to general G and discuss the geometry of the space

$$G(\mathbb{Q})\backslash G(\mathbb{A}).$$

This will be much more complicated than in the case $G = \mathrm{GL}_1$.

There exists a central subgroup $A_G \subseteq G(\mathbb{R})$, with $A_G \cong \mathbb{R}^r_{>0}$, such that

$$[G] := A_G G(\mathbb{Q})\backslash G(\mathbb{A})$$

has finite volume, cf. [GH19, Theorem 2.6.2.]. Namely, A_G can be taken as the connected component of the maximal split subtorus of the center $Z(G)$ of G, e.g., if $G = \mathrm{GL}_n$, then $A_{\mathrm{GL}_n} \subseteq \mathrm{GL}_n(\mathbb{R})$ is the group of scalar matrix with entries in $\mathbb{R}_{>0}$ while A_{SL_n} is trivial.

Again the full space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is a Hilbert space integral of the representations $L^2([G])$ over $\widehat{A_G}$, and thus considering $L^2([G])$ instead of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ reduces some analytic issues.[4]

Although we are only interested in some arbitrary right-invariant measure on $G(\mathbb{A})$[5] let us mention that there exists a canonical measure on $G(\mathbb{A})$, the Tamagawa measure, and that for this measure the number

$$\tau(G) := \mathrm{vol}([G])$$

is called the Tamagawa number of G. The knowledge of $\tau(G)$ is arithmetically interesting. Cf. [Col11, Appendix B], [Lur, Lecture 1]. For example, $\tau(\mathrm{SL}_n) = 1$, which implies $\mathrm{vol}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})) = \zeta(2)\zeta(3)\ldots\zeta(n)$, where $\zeta(s) =$ is the Riemann ζ-function.

The topological spaces $G(\mathbb{Q})\backslash G(\mathbb{A})$ and $[G]$ are profinite coverings of some real manifold, e.g., if $G = \mathrm{GL}_n$, then

$$\mathrm{GL}_n(\mathbb{Z})\backslash(\prod_p \mathrm{GL}_n(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{R})),$$

where the action of $\mathrm{GL}_n(\mathbb{Z})$ is diagonally, and thus $\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})$ covers the real manifold

$$\mathrm{GL}_n(\mathbb{Z})\backslash\mathrm{GL}_n(\mathbb{R}).$$

Similarly, $[G]$ covers $\mathrm{GL}_n(\mathbb{Z})\mathbb{R}_{>0}\backslash\mathrm{GL}_n(\mathbb{R})$. Cf. [GH19, Section 2.6], [Del73, (0.1.4.1)].

More precisely, for each compact-open subgroup $K \subseteq G(\mathbb{A}_f)$ (also called a "level subgroup") the quotient

$$[G]/K$$

is a real manifold.

Similarly, let $K_\infty \subseteq G(\mathbb{R})$ be a maximal compact (connected or not) subgroup of $G(\mathbb{R})$, e.g., if $G = \mathrm{GL}_n$, then one can take $K_\infty \cong \mathrm{SO}_n(\mathbb{R})$. As K_∞ intersects

[4]For precision, let us mention that each right-invariant Haar measue on $G(\mathbb{A})$ descents to $[G]$, cf. [GH19, Section 2.6].

[5]Because the resulting $L^2([G])$-spaces are isomorphic.

$A_G \subseteq G(\mathbb{R})$ trivially, the space $[G]$ is a principal K_∞-bundle (or K_∞-torsor) over $[G]/K_\infty$.

In the next lecture we will discuss in more detail the case $G = \mathrm{GL}_2$. For now let us just mention that if $K_\infty \cong \mathrm{SO}_2(\mathbb{R})$, then by Möbius transformations

$$A_G\backslash \mathrm{GL}_2(\mathbb{R})/K_\infty \xrightarrow{\sim} \mathbb{H}^\pm,\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g(i) := \frac{a\cdot i + b}{c \cdot i + d}$$

with

$$\mathbb{H}^\pm := \{z \in \mathbb{C} \mid \mathrm{Im}(z) \neq 0\}$$

the upper/lower halfplane. In particular,

$$[\mathrm{GL}_2]/K_\infty \cong \mathrm{GL}_2(\mathbb{Z})\backslash(\mathrm{GL}_2(\prod_p \mathbb{Z}_p) \times \mathbb{H}^\pm)$$

Let $K \subseteq G(\mathbb{A}_f)$ be a compact-open subgroup, then the set

$$G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K = \coprod_{i=1}^m G(\mathbb{Q}) g_i K$$

is finite (cf. [GH19, Theorem 2.6.1]) and

$$[G]/K_\infty K \cong \coprod_{i=1}^m \Gamma_i \backslash X,$$

where

$$X := A_G\backslash G(\mathbb{R})/K_\infty$$

is a real manifold, and

$$\Gamma_i := G(\mathbb{Q}) \cap G(\mathbb{R}) g_i K g_i^{-1} \subseteq G(\mathbb{R})$$

is a discrete subgroup (an example of a *congruence subgroup*, cf. [GH19, Section 2.6]). For K is sufficiently small, the group Γ_i acts freely and properly discontinuously on X, cf. [GH19, Definition 15.2]. In this case, the morphism

$$X \to \Gamma_i\backslash X$$

is a covering (in the sense of topology) with covering group Γ_i. The case we are mostly interested in is the case $G = \mathrm{GL}_2$. Then everything becomes more explicit. Namely,

$$\Gamma_i\backslash X \cong \Gamma\backslash \mathbb{H}^\pm$$

with $\Gamma \subseteq \mathrm{GL}_2(\mathbb{Z})$ a discrete subgroup containing

$$\Gamma(m) := \ker(\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/m))$$

for some $m \geq 0$. For $m \geq 3$, the action of $\Gamma(m)$ is free on $\mathbb{H}^\pm$. This example will provide the link with modular forms in the next lecture.

Note that for arbitrary G we in general have $\dim(\Gamma_i\backslash X) > 0$. Thus the geometry $[G]$, and the space $L^2([G])$, is *much* more complicated than in case $G = \mathbb{G}_m$.

The real manifolds $\Gamma_i\backslash X$ are interesting real manifolds, because of their arithmetic significance. We will see that if $G = \mathrm{GL}_2$, these Riemann surfaces are quasi-projective and *canonically defined over number fields.*

If the space $[G]$ is compact (which happens if and only if any $\mathbb{G}_m \subseteq G$ lies in center, cf. [GH19, Theorem 2.6.2]), the analysis of $L^2([G])$ is more simple. A concrete example, when this happens is in the case that G is the units in a non-split quaternion algebra over $\mathbb{Q}$, i.e., the group of units in a $\mathbb{Q}$-algebra with presentation

$\langle x, y | x^2 = a, y^2 = b, xy = -yx\rangle$, for suitable $a, b \in \mathbb{Q}^\times$. Note that $G(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{R})$ if $ab < 0$, and thus the $\Gamma_i\backslash X$ are still quotients of the upper half-plane. For more details on quaternion algebras see [GS17, Chapter 1].

Reductive groups over $\mathbb{Q}$ exist in abundance. A particular, easy way to produce many examples is via Weil restrictions. To introduce this let $F/\mathbb{Q}$ be a finite extension, H reductive over F. Then the functor $R \mapsto H(R \otimes_{\mathbb{Q}} F)$ on the category of $\mathbb{Q}$-algebras is representable by a reductive group

$$G := \mathrm{Res}_{F/\mathbb{Q}}(H)$$

over $\mathbb{Q}$, cf. [CGP15, Appendix A] or [GH19, Section 1.4]. For this reductive group G we get

$$G(\mathbb{Q})\backslash G(\mathbb{A}) = H(F)\backslash H(\mathbb{A}_F)$$

with $\mathbb{A}_F$ ring of adéles for F. This implies that for the Langlands program considering *all* reductive groups over $\mathbb{Q}$ is equivalent to considering all reductive groups over all number fields.

When $H = \mathbb{G}_{m,F}$ the desired decomposition of $L^2([G])$ is (roughly) equivalent to class field theory for F. Let us mention that the finiteness of the volume of $[G]$ incorporates the finiteness of the class number for F and Dirichlet's theorem on units for F.

For another source of examples consider the case when F is imaginary quadratic and $H = \mathrm{GL}_{2,F}$. Then $G(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{C})$, we can take $K_\infty \cong U(n)$ as the unitary group and

$$X = A_G\backslash G(\mathbb{R})/K_\infty$$

is the 3-dimensional hyperbolic space $\mathbb{H}^3$. The quotients $\Gamma_i\backslash X$ in this case are called arithmetic hyperbolic 3-manifolds, cf. [Thu82, Section 4 & 5]. More precisely, the hyperbolic 3-space $\mathbb{H}^3$ can be modelled on the quaternions

$$\{q = x + y \cdot i + z \cdot j \mid x, y \in \mathbb{R},\ z > 0\}$$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ acts on $\mathbb{H}^3$ via $q \mapsto (aq + b)(cq + d)^{-1}$, where $\mathbb{C} = \{x + y \cdot i \mid x, y \in \mathbb{R}\}$.

After having discussed some examples, we can now give the definition of an automorphic representation (in the L^2-sense), which is of central importance in the Langlands program.

Definition 2.4 ([GH19, Definition 3.3])**.** *Let G be a reductive group over $\mathbb{Q}$. An automorphic representation (in the L^2-sense) is an irreducibe unitary $G(\mathbb{A})$-representation π which is isomorphic to a subquotient of $L^2([G])$.*

Later, we will also discuss a different, more algebraic notion of an automorphic representation. In the above definition, π is necessarily trivial on A_G. This assumption is usually harmless. By Schur's lemma for irreducible, unitary representations we can find for any irreducible, unitary $G(\mathbb{A})$-representation π a character $\chi\colon G(\mathbb{R}) \to \mathbb{C}^\times$ with $\chi \otimes \pi$ trivial on A_G, cf. [GH19, Section 6.5]. E.g., for $G = \mathrm{GL}_n$ one can take χ of the form $g \mapsto |\det(g)|^s)$ for some $s \in \mathbb{R}_{>0}$.

The space $L^2([G])$ decomposes into a "discrete" and a "continuous" part, cf. [GH19, Section 10.4]. From an analytic perspective the case that $[G]$ is compact is easier as there the continuous part vanishes.

We now explain more precisely what is meant by the "discrete" and "continuous" part. Recall that $\widehat{G(\mathbb{A})}$ denotes the set of isomorphism classes of irreducible unitary $G(\mathbb{A})$-representations. Generalizing the compact-open topology on $\widehat{A}$ for A a locally compact abelian group, the set $\widehat{G}$ has a natural topology, the Fell topology, cf. [GH19, Section 3.8]. Roughly, the Fell topology can be described as follows. Consider $\pi \in \widehat{G(\mathbb{A})}$, $x \in \pi$, and $\varphi\colon \pi \to \mathbb{C}$ a continuous $\mathbb{C}$-linear homomorphism. Then

$$f_{x,\varphi}\colon G \to \mathbb{C},\ g \mapsto \varphi(gx)$$

is called a matrix coefficient of π. Roughly, two $\pi, \pi' \in \widehat{G(\mathbb{A})}$ are close in the Fell topology if their matrix coefficients are close for the compact-open topology.

The following theorem holds more generally for a unitary representation of $G(\mathbb{A})$. We don't need the exact meaning of the integral in Theorem 2.5, let us just mention that its occurence is due to the fact that some representations of $G(\mathbb{A})$ "contribute" to $L^2([G])$ without being subrepresentations (similar to the case of Fourier theory for $\mathbb{R}$).

Theorem 2.5 ([GH19, Theorem 3.9.4]). *There exists a measurable multiplicity function* $m\colon \widehat{G} \to \{1, 2, \ldots, \infty\}$ *and a measure* μ *on* $\widehat{G}$ *such that*

$$L^2([G]) \cong \int_{\widehat{G}} (\widehat{\bigoplus}^{m(\pi)} \pi) d\mu(\pi).$$

The m *and* μ *are unique up changes on sets of measure* 0.

With this theorem we can now explain roughly what is by the "discrete" and "continuous" part of $L^2([G])$. Points $\pi \in \widehat{G(\mathbb{A})}$ with $\mu(\pi) > 0$ are "discrete" for the measure m on $\widehat{G(\mathbb{A})}$, and they appear as *sub*representations in $L^2([G])$ with multiplicty $m(\pi)$. This motivates the decomposition

$$L^2([G]) \cong L^2_{\mathrm{disc}}([G]) \oplus L^2_{\mathrm{cont}}([G]),$$

where $L^2_{\mathrm{disc}}([G])$ is the Hilbert sum of all irreducible, unitary *sub*representations of $G(\mathbb{A})$, and $L^2_{\mathrm{cont}}([G])$ its orthogonal complement, cf. [GH19, Section 9.1].

In a pioneering work Langlands described the continuous part via (proper) parabolic subgroups of G. Recall that by definition, a closed, connected subgroup $P \subseteq G$ (over $\mathbb{Q}$) is parabolic if G/P is a proper scheme over $\mathrm{Spec}(\mathbb{Q})$, and that the quotient of P by its unipotent radical is its Levi quotient M, cf. [GH19, Section 1.9]. For example, for GL_n take a decomposition $n = n_1 + n_2 + \ldots + n_k$, and P as the subgroup of block upper triangular matrices in GL_n with blocks of size $n_1, \ldots, n_k$. The Levi quotient is $M =$ block diagonal matrices. Up to conjugation all parabolics in GL_n are of this form.

Langlands then proved that the continuous part $L^2_{\mathrm{cont}}([G])$ can be described via "inducing" the discrete parts $L^2_{\mathrm{disc}}([M])$ for all Levi quotients of parabolics $P \subseteq G$, $P \neq G$, cf. [GH19, Section 10.4]). This is his theory of "Eisenstein series", and it is a starting point of the Langlands program.

For this course, $L^2_{\mathrm{disc}}([G])$ (actually the proper "cuspidal" subspace) will be more important than the whole space $L^2([G])$. Let us mention that $[G]$ is compact if and only G has no proper parabolics (defined over $\mathbb{Q}$). This is in accordance with the previous claim that $L^2([G]) = L^2_{\mathrm{disc}}([G])$ in this case.

3. From modular forms to automorphic representations, part I

Starting with this lecture we want to construct automorphic representations for GL_2 associated to modular forms.

Let us recall that we discussed last time:

- Consider a G reductive group over $\mathbb{Q}$, and $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ the ring of adéles.
- Then we introduce the space $[G] = A_G G(\mathbb{Q})\backslash G(\mathbb{A})$, called the adélic quotient, which has finite $G(\mathbb{A})$-invariant volume.
- We stated that the Langlands program aims to decompose the unitary $G(\mathbb{A})$-representation
$$L^2([G])$$
according to arithmetic data.
- In the example, $G = \mathbb{G}_m$ we showed that
$$L^2(\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times) \cong \widehat{\bigoplus_{\sigma\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\mathbb{C}^\times}} \mathbb{C}\circ\chi_\sigma,$$
as an example that the desired arithmetic data arise from Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
- Then we discussed the geometry of $[G]$:
  - let $K_\infty \subseteq G(\mathbb{R})$ be some maximal compact, connected subgroup,
  - let $K \subseteq G(\mathbb{A}_f)$ be some compact-open subgroup.
  - Then

$$\begin{array}{ll}
[G] & \\
\Big\downarrow\ K\text{-torsor (in particular: a profinite covering)} & \\
[G]/K & \text{is a real manifold, at least for sufficiently small } K \\
\Big\downarrow\ K_\infty\text{-torsor} & \\
[G]/KK_\infty = \coprod_{i=1}^{m}\Gamma_i\backslash X & \text{is a disjoint union of arithmetic manifolds}
\end{array}$$

  with $\Gamma_i \subseteq G(\mathbb{R})$ some (discrete) congruence subgroups determined by K, and
$$X := A_G\backslash G(\mathbb{R})/K_\infty.$$
- Moreover, we defined that an irreducible unitary representation π of $G(\mathbb{A})$ is automorphic (in the L^2-sense) if it is isomorphic to a subquotient of $L^2([G])$.
- First examples (only mentioned implicitly) are the following:
  - if $G = \mathbb{G}_m$, then the automorphic representations (in the L^2-sense) of $G(\mathbb{A})$ are the unitary characters of $\mathbb{A}^\times$, which are trivial on $\mathbb{Q}^\times\cdot\mathbb{R}_{>0}$.
  - if G is general the trivial representations (=constant functions on $[G]$) is automorphic because $[G]$ has finite volume. Moreover, it lies in the discrete part $L^2_{\mathrm{disc}}([G])$ of $L^2([G])$.
- For general G there exists the orthogonal decomposition
$$L^2([G]) \cong L^2_{\mathrm{disc}}([G]) \oplus L^2_{\mathrm{cont}}([G])$$
with $L^2_{\mathrm{disc}}([G])$ the closure of the sum of all irreducible *sub*representations of $G(\mathbb{A})$.

- Finally, we mentioned that Langlands has described $L^2_{\mathrm{cont}}([G])$ via $L^2_{\mathrm{disc}}([M])$ for M running through Levi quotients of parabolic subgroups $P \subseteq G, P \neq G$ (which are defined over $\mathbb{Q}$).

Today, we will discuss in more detail the case $G = \mathrm{GL}_2$. First of all,

$$A_{\mathrm{GL}_2} \cong \mathbb{R}_{>0}$$

embedded as scalar matrices into $\mathrm{GL}_2(\mathbb{R})$. Last time we claimed that we have

$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}) \cong \mathrm{GL}_2(\mathbb{Z})\backslash(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \times \mathrm{GL}_2(\mathbb{R})),$$

where

$$\widehat{\mathbb{Z}} := \varprojlim_m \mathbb{Z}/m \cong \prod_p \mathbb{Z}_p.$$

Let us explain why this is true, cf. [Tho, Theorem 1]. The group $\mathrm{GL}_2(\mathbb{A}_f)$ is the restricted product of the groups $\mathrm{GL}_2(\mathbb{Q}_p)$ with respect to the compact-open subgroups $\mathrm{GL}_2(\mathbb{Z}_p)$, i.e.,

$$\mathrm{GL}_2(\mathbb{A}_f) = \{(A_p)_p \in \prod_p \mathrm{GL}_2(\mathbb{Q}_p) \mid A_p \in \mathrm{GL}_2(\mathbb{Z}_p) \text{ for all but finitely many } p\}$$

(cf. [GH19, Proposition 2.3.1]). The Chinese remainder theorem implies that $\mathbb{Q} \subseteq \mathbb{A}_f$ (embedded diagonally) is dense. Moreover,

$$\mathbb{A}_f^\times = \mathbb{Q}^\times \cdot \widehat{\mathbb{Z}}^\times$$

using prime factorization.

The groups $\mathrm{GL}_2(\mathbb{Z}_p), \mathrm{GL}_2(\mathbb{Q}_p)$ are generated by elementary and diagonal matrices which implies that

$$\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2(\widehat{\mathbb{Z}}),$$

and thus

$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f) \cong \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{GL}_2(\widehat{\mathbb{Z}}) \cong \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\widehat{\mathbb{Z}}).$$

From here we can deduce that as desired

$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}) \cong \mathrm{GL}_2(\mathbb{Z})\backslash(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \times \mathrm{GL}_2(\mathbb{R})).$$

The proof works in fact for all $n \geq 1$ and shows

$$\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A}) \cong \mathrm{GL}_n(\mathbb{Z})\backslash(\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \mathrm{GL}_n(\mathbb{R})).$$

Let us define

$$K_m = \ker(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/m))$$

for $m \geq 1$. Then these groups are cofinal within all compact-open subgroups in $\mathrm{GL}_2(\mathbb{A}_f)$. We have

$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f)/K_m \cong \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\widehat{\mathbb{Z}})/K_m$$

and $\mathrm{GL}_2(\widehat{\mathbb{Z}})/K_m \cong \mathrm{GL}_2(\mathbb{Z}/m)$. The group $\mathrm{SL}_2(\mathbb{Z}/m)$ is even generated by elementary matrices.[6] This implies that the morphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/m)$ is surjective. Thus, we obtain

$$\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\widehat{\mathbb{Z}})/K_m \cong \{\pm 1\}\backslash(\mathbb{Z}/m)^\times$$

---

[6] use the euclidean algorithm and the magic identity

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for any invertible element a in some ring R.

via the determinant. Now, define

$$K_\infty := \mathrm{SO}_2(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{R})$$

and

$$X := A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R})/K_\infty.$$

Then $X \cong \mathbb{H}^\pm$ (the upper/lower halfplane) via the morphsim

$$A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R})/K_\infty \xrightarrow{\sim} \mathbb{H}^\pm,\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g(i) := \frac{a\cdot i + b}{c \cdot i + d}.$$

Thus, we obtain in the end

$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K_m \times X) \cong \coprod_{\{\pm 1\}\backslash(\mathbb{Z}/m)^\times} \Gamma(m)\backslash \mathbb{H}^\pm$$

with

$$\Gamma(m) := K_m \cap \mathrm{GL}_2(\mathbb{Z}) = \ker(\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/m)).$$

The fact that $\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K_m \times X)$ is non-connected may seem annoying, but it will turn out to be an advantage. Note that if we replace $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ by some compact-open subgroup K such that $\det\colon K \to \widehat{\mathbb{Z}}^\times$ is surjective, e.g., K equals

$$K_1(m) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{A}_f) \mid c \equiv 0, d \equiv 1 \bmod m, \},$$

then $\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K_m \times X)$ is connected.

The finiteness of the volume of $[G]$ for $G = \mathrm{GL}_2$ can now be checked via a direct argument.

The compactness of $K_\infty K_m$ implies that it suffices to show that the space $[\mathrm{GL}_2]/K_\infty K_m$ has finite volume. Writing this space as a disjoint union we can even reduce to the statement that $\Gamma(m)\backslash \mathbb{H}^\pm$ has finite volume for $m \geq 1$. Up to scalar in $\mathbb{R}_{>0}$, the $\mathrm{GL}_2(\mathbb{R})$-invariant measure on $\mathbb{H}^\pm$ is given by the volume form

$$\frac{1}{y^2} dx \wedge dy.$$

Indeed, we have to show

$$g^*(\frac{1}{y^2} dx \wedge dy) = \frac{1}{y^2} dx \wedge dy$$

for $g \in \mathrm{GL}_2(\mathbb{R})$. Now, write $dx \wedge dy = \frac{i}{2} dz \wedge d\overline{z}$ with $z = x + iy$. The statement can now easily shown to be true for

$$g = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad a \in \mathbb{R},$$

for A_{GL_2}, for

$$g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad a \in \mathbb{R}^\times,$$

and finally

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

i.e., for $z \mapsto \frac{1}{z} = \frac{\overline{z}}{|z|^2}$. From the magic identity mentioned before we can conclude invariance for lower triangular matrices, and thus for all of $\mathrm{GL}_2(\mathbb{R})$.

Let us now show that $\Gamma(m)\backslash\mathbb{H}^{\pm}$ has finite volume. A known fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ is given by the set

$$D := \{z \in \mathbb{H} \mid |z| > 1, |\mathrm{Re}(z)| < 1/2\}$$

and finiteness of the volume of $\mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^{\pm}$ follows from

$$\int\limits_D \frac{1}{y^2} dx \wedge dy < \infty.$$

Finally, as $\Gamma(m) \subseteq \mathrm{GL}_2(\mathbb{Z})$ is of finite index $\Gamma(m)\backslash\mathbb{H}^{\pm}$ is of finite volume, as desired.

The Langlands program incorporates in particular a close connection between modular forms and 2-dimensional (ℓ-adic) Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Discussing this connection will be a major theme of the course. First let us recall the classical definition of modular forms, cf. [DS05]. For this, let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, later assumed to be sufficiently small. Fix $k \in \mathbb{Z}$.

Definition 3.1. *A function $f \colon \mathbb{H}^{\pm} \to \mathbb{C}$ is a modular form of weight k for Γ if*

- $f(\gamma \cdot z) = (cz + d)^k f(z)$ *for* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$
- f *is holomorphic*
- f *is holomorphic at the cusps of* Γ *(we will explain this in a second).*

Let $M_k(\Gamma)$ be the $\mathbb{C}$-vector space of modular forms of weight k for Γ. Let us also call a function weakly modular if it satisfies only the first two conditions, but is not necessarily holomorphic at the cusps.

Let us recall that the upper halfplane $\mathbb{H}$ naturally emdeds $\mathrm{SL}_2(\mathbb{Q})$-equivariantly into the set

$$\mathbb{H}^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$$

equipped with the Satake topology. The orbits of a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ are called the cusps for Γ, and the quotient

$$\Gamma\backslash\mathbb{H}^*$$

is a natural compactification of $\Gamma\backslash\mathbb{H}$. Adding another copy of $\mathbb{P}^1(\mathbb{Q})$ to the lower halfplane $\mathbb{H}^-$, one obtains similar objects/notions for $\Gamma \subseteq \mathrm{GL}_2(\mathbb{Z})$.

The condition of being holomorphic at the cusps, means now the following. For each cusp σ a weakly modular form f for Γ has a Fourier expansion, i.e., up to translating the cusp to $\infty \in \mathbb{P}^1(\mathbb{Q})$ there exists an equality

$$f(z) = \sum_{n \in \mathbb{Z}} a_n q^{n/h}$$

for $q = e^{2\pi i z}$, h the so-called smallest periodicity of f, and $a_n \in \mathbb{C}$, $n \in \mathbb{Z}$. The weakly modular form f is then called holomorphic at this cusp if $a_n = 0$ for $n < 0$, and it is called holomorphic at the cusps if it is holomorphic at every cusp. If furthermore $a_0 = 0$ for each cusp, then f is called a cusp form. We denote by

$$S_k(\Gamma) \subseteq M_k(\Gamma)$$

the subspace of cusp forms of weight k for Γ.

Our next aim is to describe (weakly) modular forms of weight $k \in \mathbb{Z}$ for $\Gamma \subseteq \mathrm{GL}_2(\mathbb{Z})$ canonically as sections of some holomorphic line bundle

$$\omega^k$$

over $\Gamma\backslash\mathbb{H}^{\pm}$ (at least for sufficiently small Γ).

Before doing this we discuss some useful constructions of "descent for fiber bundles", which will be used frequently.

Consider a set Z, a group H acting freely on Z (from the right). Then the projection

$$Z \to Z/H$$

deserves to be called an H-torsor. Let F be a set with a (left) action of H. The contracted product

$$Z \times^H F$$

of Z over H with F is defined as the quotient of

$$Z \times F$$

for the action $h \cdot (z, f) \mapsto (zh^{-1}, hf)$ of H. Via the natural projection $\overline{(z,f)} \mapsto \overline{z}$ we obtain the "fiber bundle"

$$Z \times^H F \to Z/H$$

over Z/H whose fibers are (non-canonically) isomorphic with F. Indeed, for a class $c \in Z/H$ *and* the choice of $z \in Z$ mapping to c, we obtain the isomorphism

$$F \cong (Z \times^H F) \times_{Z/H} c, \ f \mapsto (z, f).$$

Be aware that $- \times^H -$ denotes the contracted product, while $- \times_{Z/H} -$ the fiber product (and these are two unrelated constructions). In particular, there exists the *canonical* isomorphism

$$Z \times F \cong (Z \times^H F) \times_{Z/H} Z, \ (z, f) \mapsto (\overline{(z, f)}, z),$$

i.e., the fiber bundle $Z \times^H F$ is canonically trivial after pullback to Z/H.

We can conveniently describe sections of the fiber bundle $Z \times^H F \to Z/H$.

Lemma 3.2. *The set of sections $s\colon Z/H \to Z \times^H F$, i.e., morphism over Z/H, identifies canonically with the set of functions $\varphi\colon Z \to F$ such that $\varphi(zh) = h^{-1}\varphi(z)$ for all $z \in Z, h \in H$.*

Proof. Given $s\colon Z/H \to Z \times^H F$ and $z \in Z$, there exists (by freeness of the action of H on Z) a unique element $f \in F$ such that

$$s(\overline{z}) = \overline{(z, f)}.$$

Set $\varphi(z) := f$. Then

$$\overline{(z, \varphi(z))} = s(\overline{z}) = s(\overline{zh}) = \overline{(zh, \varphi(zh))} = \overline{(z, h\varphi(zh)},$$

which implies

$$\varphi(z) = h\varphi(zh)$$

as desired. Conversely, given as $\varphi\colon Z \to F$ satisfying $\varphi(zh) = h^{-1}\varphi(z)$ for all $z \in Z, h \in H$, the function

$$Z \to Z \times F, \ z \mapsto (z, \varphi(z))$$

is, by definition, invariant for the action of H, and thus descents to a section

$$s\colon Z/H \to Z \times^H F.$$

Both constructions are inverse to each other. ∎

Conversely, assume we are given any morphism $Y \to Z/H$ with an isomorphism over Z

$$g\colon Y \times_{Z/H} Z \cong Z \times F$$

for some set F. Pick $z \in Z$ with image $c \in Z/H$. Then for any $h \in H$ we obtain an isomorphism

$$\sigma_h\colon F \overset{g^{-1}_{|z\times F}}{\cong} Y_c \times_c z \overset{h}{\cong} Y_c \times_c zh \overset{g_{|g^{-1}(zh\times F)}}{\cong} F,$$

which define an action of H on F. Moreover, one can check that via this action

$$Y \cong Z \times^H F.$$

In more fancy terms, one obtains that "fiber bundles" $Y \to Z/H$ with a "trivialization" over Z are equivalent to sets with an action of H.

Let us mention also the following variant, when $Z = G$ is a group and $H \subseteq G$ a subgroup. A G-equivariant set Y over G/H is a space $\pi\colon Y \to G/H$ over G/H together with a left action of G such that π is G-equivariant. In this case, one has the canonical trivialization

$$G \times Y_{1\cdot H} \cong Y \times_{G/H} G,\ (g, y) \mapsto (gy, g)$$

and one can conclude that the category of G-equivariant sets over G/H is just equivalent to the category of sets equipped with a (left) action of H, because for an H-set F the contracted product

$$G \times^H F$$

is naturally G-equivariant (by letting G via left multiplication on the left factor).

Note that we required that H acts on Z (resp. G) on the right, but of course we can similarly consider "fiber bundles" over $H\backslash Z$.

We placed the constructions in the category of sets, but it is clear that if Z, H, F are topological spaces (real manifolds, complex manifolds,...) and the action of H is continuous (smooth, holomorphic, ...) we can perform the analogous constructions in the category of topological spaces (real manifolds, complex manifolds,...) etc..

Let us now start to describe modular forms as sections of line bundles on modular curves.

First of all, let us note that the line bundles

$$\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(k)$$

for $k \in \mathbb{Z}$ are naturally $\mathrm{GL}_2(\mathbb{C})$-equivariant. Indeed, by taking tensor powers it is enough to consider the case $k = 1$. But then $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ is associated to the natural $\mathbb{G}_m$-torsor

$$\mathbb{A}^2_{\mathbb{C}} \setminus \{0\} \to \mathbb{P}^1_{\mathbb{C}},\ (z_1, z_2) \mapsto [z_1 : z_2],$$

which is naturally $\mathrm{GL}_2(\mathbb{C})$-equivariant. Next, we observe that the embedding

$$\mathbb{H}^{\pm} \subseteq \mathbb{P}^1_{\mathbb{C}},\ z \mapsto [z : 1]$$

is $\mathrm{GL}_2(\mathbb{R})$-equivariant, and thus we obtain for $n \in \mathbb{Z}$ the $\mathrm{GL}_2(\mathbb{R})$-equivariant line bundle

$$\begin{array}{c} \mathcal{O}_{\mathbb{H}^{\pm}}(k) := \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(k) \times_{\mathbb{P}^1_{\mathbb{C}}} \mathbb{H}^{\pm} \\ \downarrow \\ \mathbb{H}^{\pm} \end{array}$$

over $\mathbb{H}^\pm$. The stabilizer of $i \in \mathbb{H}^\pm$ in $\mathrm{GL}_2(\mathbb{R})$ identifies with

$$\mathbb{C}^\times \cong \mathbb{R}_{>0}\mathrm{SO}_2(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{R}).$$

Using

$$\mathbb{H}^\pm \cong \mathrm{GL}_2(\mathbb{R})/\mathbb{C}^\times$$

the $\mathcal{O}_{\mathbb{H}^\pm}(k)$ correspond, under the constructions for fiber bundles mentioned before, to the (holomorphic) representation

$$\chi_n \colon \mathbb{C}^\times \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times,\ z \mapsto z^k.$$

Note that Lemma 3.2 implies that *smooth* sections of $\mathcal{O}_{\mathbb{H}^\pm}(k)$ on $\mathbb{H}^\pm$ identify with smooth functions

$$\varphi \colon \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}$$

satisfying

$$\varphi(gz) = z^{-k}\varphi(g), \quad g \in \mathrm{GL}_2(\mathbb{R}), z \in \mathbb{C}^\times.$$

This will be used later. From now on we assume that Γ is sufficiently small, i.e., it acts freely on $\mathbb{H}^\pm$. Modding out the action of Γ on $\mathcal{O}_{\mathbb{H}^\pm}(k)$ yields the holomorphic line bundle

$$\begin{array}{c} \omega^{\otimes k} := \Gamma\backslash\mathcal{O}_{\mathbb{H}^\pm}(k) \\ \big\downarrow \\ \Gamma\backslash\mathbb{H}^\pm, \end{array}$$

over $\Gamma\backslash\mathbb{H}^\pm$. To get the link of sections of ω^k with (weakly) modular forms of weight k for Γ, we have to make the $\mathrm{GL}_2(\mathbb{R})$-action on

$$\mathcal{O}_{\mathbb{H}^\pm}(k)$$

more explicit. For $k \in \mathbb{Z}$ we define the holomorphic $\mathrm{GL}_2(\mathbb{R})$-equivariant line bundle

$$L_k := \mathbb{H}^\pm \times \mathbb{C}$$

over $\mathbb{H}^\pm$ with $\mathrm{GL}_2(\mathbb{R})$-acting on the left by

$$g \cdot (z, \lambda) := (\frac{az+b}{cz+d}, (cz+d)^k\lambda)$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$. For notational convenience let us introduce the function (a "factor of automorphy")

$$j \colon \mathrm{GL}_2(\mathbb{R}) \times \mathbb{H}^\pm \to \mathbb{C}^\times,\ (g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \mapsto j(g,z) := cz+d.$$

We leave it as an exercise to check that he map

$$\begin{array}{l} L_1^* \to \mathcal{O}_{\mathbb{H}^\pm}(1)^* = \mathbb{H}^\pm \times_{\mathbb{P}^1_{\mathbb{C}}} \mathbb{A}^2_{\mathbb{C}} \setminus \{0\}, \\ (z,\lambda) \mapsto (z, (\lambda z, \lambda)) \end{array}$$

is a holomorphic isomorphism of $\mathrm{GL}_2(\mathbb{R})$-equivariant line bundles over $\mathbb{H}^\pm$. Here (and in the following) $(-)^*$ denotes the complement of the zero section in a line bundle. By taking tensor powers we obtain

$$L_k \cong \mathcal{O}_{\mathbb{H}^\pm}(k)$$

for all $k \in \mathbb{Z}$, as $\mathrm{GL}_2(\mathbb{R})$-equivariant holomorphic line bundles.[7] Similarly to Lemma 3.2[8] we get that a holomorphic section

$$\mathbb{H}^{\pm} \to L_k,\ z \mapsto (z, f(z))$$

is Γ-equivariant if and only if $f(z)$ satisfies

$$f(\gamma \cdot z) = j(\gamma, z)^k f(z)$$

for $\gamma \in \Gamma$, i.e.,

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$$

if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. In other words, we proved that

$$H^0(\Gamma\backslash\mathbb{H}^{\pm}, \omega^{\otimes k})$$

identifies with the space of weakly modular forms for Γ. In particular, we have a natural embedding

$$M_k(\Gamma) \subseteq H^0(\Gamma\backslash\mathbb{H}^{\pm}, \omega^{\otimes k})$$

with image defined by condition of being "holomorphic at the cusps". Let us analyze what the condition of being holomorphic at the cusps means in terms of sections of ω^k.

We claim that the ω^k extend canonically to the (Satake) compactification

$$\overline{\Gamma\backslash\mathbb{H}^{\pm}}$$

of $\Gamma\backslash\mathbb{H}^{\pm}$. The stabilizer Γ_∞ of the cusp $\infty \in \mathbb{P}^1(\mathbb{Q})$ in $\mathrm{SL}_2(\mathbb{Z})$ is the subgroup of elements

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

with $a \in \mathbb{Z}$. Thus, it suffices to see that ω extends (canonically) along the embedding

$$\Gamma_\infty\backslash(\mathbb{H} \cup \{\infty\}).$$

Write

$$L_k^+ := L_{k|\mathbb{H}}.$$

Then the Γ_∞-action on $L_k^+ = \mathbb{H} \times \mathbb{C}$ is only via the first factor (as $j(\gamma, z) = z$ for all $z \in \mathbb{H}$ and any $\gamma \in \Gamma_\infty$). Note that via expontial map

$$\Gamma_\infty\backslash\mathbb{H} \cong \mathbb{D}^\times := \{q \in \mathbb{C}^\times \mid 0 < |q| < 1\}$$

(this was implicitly used when discussing Fourier expansions). Thus canonically,

$$\Gamma_\infty\backslash L_k^+ \cong \mathbb{D}^\times \times \mathbb{C},$$

which clearly extends to $\mathbb{D} := \{q \in \mathbb{C}^\times \mid |q| < 1\}$, namely to $\mathbb{D} \times \mathbb{C}$. This proves as desired that ω^k extends canonically to the mentioned (Satake) compactification $\overline{\Gamma\backslash\mathbb{H}^{\pm}}$ of $\Gamma\backslash\mathbb{H}^{\pm}$. We called this extension again ω^k. The condition of holomorphicity

[7]That L_k and $\mathcal{O}_{\mathbb{H}^{\pm}}(k)$ are abstractly isomorphic, as $\mathrm{GL}_2(\mathbb{R})$-equivariant line bundles can also be deduced from the observation that both correspond to the $\mathbb{C}^\times$-representation $z \mapsto z^k$. This was noted for $\mathcal{O}_{\mathbb{H}^{\pm}}$, and follows because under the embedding $\mathbb{C}^\times$ an element $z \in \mathbb{C}^\times$ is mapped to the matrix $g = \begin{pmatrix} \mathrm{Re}(z) & -\mathrm{Im}(z) \\ \mathrm{Im}(z) & \mathrm{Re}(z) \end{pmatrix}$ and $j(g, i) = z$.

[8]The sitation is slightly different than in Lemma 3.2, namely, these Γ-equivariant vector bundles are not associated to a representation of Γ.

at the cusps now translates into the statement that the section $f \in H^0(\Gamma\backslash\mathbb{H}^\pm, \omega^k)$ extends to a section of ω^k on $\overline{\Gamma\backslash\mathbb{H}^\pm}$.

Using a bit more theory, the Riemann surface

$$\Gamma\backslash\mathbb{H}^\pm$$

is an algebraic curve over $\mathbb{C}$, and the analytification of its canonical compactification is $\overline{\Gamma\backslash\mathbb{H}^\pm}$. As $M_k(\Gamma)$ is the image of $H^0(\overline{\Gamma\backslash\mathbb{H}^\pm}, \omega^k)$ in $H^0(\Gamma\backslash\mathbb{H}^\pm, \omega^k)$, one can deduce (from GAGA) that $M_k(\Gamma)$ is finite-dimensional over $\mathbb{C}$. Moreover, it can be shown that ω is ample on $\overline{\Gamma\backslash\mathbb{H}^\pm}$ which implies that $M_k(\Gamma) = 0$ for $k < 0$ and $M_k(\Gamma)$ gets "big" for $k \gg 0$. In fact, using the theorem of Riemann Roch one can calculate the dimension of $M_k(\Gamma)$. Let us finally note that

$$S_k(\Gamma) \cong H^0(\overline{\Gamma\backslash\mathbb{H}^\pm}, \omega^{\otimes k}(-D))$$

where $D = \overline{\Gamma\backslash\mathbb{H}^\pm} \setminus \Gamma\backslash\mathbb{H}^\pm$ is the (reduced) divisor at infinity.

For more details, we refer to [DS05, Chapter 3, Chapter 7].

We now explain how to pass from sections of $\omega^{\otimes k}$ to functions on $\mathrm{GL}_2(\mathbb{A})$ (or $[\mathrm{GL}_2]$).

For this we will interpret modular forms for Γ with $\Gamma \subseteq \mathrm{GL}_2(\mathbb{Z})$ varying among all congruence subgroup via "modular forms" for K where $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ is varying among all compact-open subgroups.

Let $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ be a compact-open subgroup and write

$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm) \cong \coprod_{i=1}^m \Gamma_i\backslash\mathbb{H}^\pm$$

with $\Gamma_i \subseteq \mathrm{GL}_2(\mathbb{Z})$ some congruence subgroups. Then $\mathcal{O}_{\mathbb{H}^\pm}(k)$ defines by pullback from $\mathbb{H}^\pm$ a (complex) line bundle $\omega^{\otimes k}$ on

$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm)$$

(we implicitly assumed that K is sufficiently small and used $\mathrm{GL}_2(\mathbb{Q})$-equivariance of $\mathcal{O}_{\mathbb{H}^\pm}(k)$).

The spaces $M_k(\Gamma_i)$ of modular forms for Γ_i, $i = 1, \ldots, m$, (even the weakly modular forms) embed into the space of holomorphic sections

$$H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm), \omega^{\otimes k}).$$

Set

$$M_k(K) := \bigoplus_{i=1}^m M_k(\Gamma_i).$$

Then $M_k(K) \subseteq H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm), \omega^{\otimes k})$ is defined analogously by the condition of being holomorphic at all cusps. We can even get rid of K and use the pullback of $\omega^{\otimes k}$ to define a $G(\mathbb{A}_f)$-equivariant line bundle, again written $\omega^{\otimes k}$, on

$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm).$$

Note that this space is no longer a complex manifold, only an inverse limit of complex manifolds along finite covering maps. But still we can define an analog of "holomorphic sections" of $\omega^{\otimes k}$ on it via

$$\begin{aligned} & H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm), \omega^{\otimes k}) \\ := \; & \varinjlim_K H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm), \omega^{\otimes k}). \end{aligned}$$

Moreover, we get the (big space)

$$M_k := \varinjlim_K M_k(K)$$

of modular forms of level k. Later we will explain that for $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ (sufficiently small) we obtain $M_k(K)$ as the invariants of K under some canonical action of K on M_k.

From our discussion of fiber bundles around Lemma 3.2 we know that the pull-back of

$$\mathcal{O}_{\mathbb{H}^\pm}(k)$$

along

$$\mathrm{GL}_2(\mathbb{R}) \to \mathbb{H}^\pm, g \mapsto g \cdot i$$

is canonically trivial (as a $\mathrm{GL}_2(\mathbb{R})$-equivariant bundle), and that *smooth* sections of $\mathcal{O}_{\mathbb{H}^\pm}(k)$ on $\mathbb{H}^\pm$ identify canonically with smooth functions

$$\varphi\colon \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}$$

satisfying

$$\varphi(gz) = z^{-k}\varphi(g), \quad g \in \mathrm{GL}_2(\mathbb{R}), z \in \mathbb{C}^\times.$$

Concretely we associate to a smooth section

$$\mathbb{H}^\pm \to L_k = \mathbb{H}^\pm \times \mathbb{C},\ z \mapsto (z, f(z))$$

the smooth function

$$\varphi_f\colon \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C},\ g \mapsto j(g,i)^{-k} f(gi).$$

Note that smooth sections of L_k on $\mathbb{H}^\pm$ are simply smooth functions on $\mathbb{H}^\pm$ (but this identification is not $\mathrm{GL}_2(\mathbb{R})$-equivariant!).

We can do something similar with $\mathbb{H}^\pm$ replaced by $\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm$, if we slightly extend the notion smoothness.

Definition 3.3. *Let $Y_i, i \in I$, a cofiltered inverse system of real manifolds such that each transition map $Y_i \to Y_j$ is a finite covering. Set $Y := \varprojlim_{i \in I} Y_i$ and let $\pi_i\colon Y \to Y_i$ be the canonical projection. Then we call a function*

$$\varphi\colon Y \to \mathbb{C}$$

smooth if $\varphi = \varphi_i \circ \pi_i$ for some $i \in I$ and some smooth function $\varphi_i\colon Y_i \to \mathbb{C}$.

Note that by the same pattern we can also define holomorphic functions (or sections of line bundles) on Y if the Y_i is additionally assumed to be a complex manifold. Of course, we want to apply this terminology to $\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm$ (or $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{R})$), which is the inverse limit of the complex manifolds $\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm$ (resp. real manifolds $\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathrm{GL}_2(\mathbb{R})$) where K is running through the cofiltered system of compact-open subgroups of $\mathrm{GL}_2(\mathbb{A}_f)$.

We obtain that smooth sections

$$\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm \to \mathrm{GL}_2(\mathbb{A}_f) \times L_k,\ (g,z) \mapsto (g, z, f(g,z))$$

identify with smooth functions

$$\varphi\colon \mathrm{GL}_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}$$

satisying

$$\varphi(g, g_\infty z) = z^{-k}\varphi(g, g_\infty)$$

for $(g, g_\infty) \in \mathrm{GL}_2(\mathbb{A})$ and $z \in \mathbb{C}^\times \subseteq \mathrm{GL}_2(\mathbb{R})$. Concretely, for f we set

$$\varphi_f(g, g_\infty) := j(g_\infty, i)^{-k} f(g, g_\infty i).$$

Note that the section

$$\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm \to \mathrm{GL}_2(\mathbb{A}_f) \times L_k,\ (g, z) \mapsto (g, z, f(g, z))$$

is invariant under the action of $\mathrm{GL}_2(\mathbb{Q})$ from the left if and only if f satisfies the modularity condition

$$f(\gamma g, \gamma z) = j(\gamma, z)^k f(g, z).$$

Using the equation

$$j(\gamma g_\infty, i) = j(\gamma, g_\infty i) j(g_\infty, i)$$

we obtain that f defines a $\mathrm{GL}_2(\mathbb{Q})$-equivariant section of L_k if and only if the function φ_f is $\mathrm{GL}_2(\mathbb{Q})$-invariant, i.e.,

$$\varphi_f(\gamma g, \gamma g_\infty) = \varphi_f(g, g_\infty)$$

for $(g, g_\infty) \in \mathrm{GL}_2(\mathbb{A})$.

Restricting to holomorphic sections we have thus constructed the natural morphism

$$H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm), \omega^{\otimes k}) \to C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$$

such that $f(g, z) \mapsto \varphi_f(g, g_\infty) := j(g_\infty, i)^{-k} f(g, g_\infty i)$, where we viewed a section of $\omega^{\otimes k}$ on $\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm)$ as a function

$$\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm \to \mathbb{C},\ (g, z) \to f(g, z)$$

satisfying the above modularity condition.

Let us recall that

$$M_k = \varinjlim_K M_k(K)$$

with $M_k(K)$ a sum of spaces of modular forms

$$\bigoplus_{i=1}^m M_k(\Gamma_i)$$

for various congruence subgroups $\Gamma_i \subseteq \mathrm{GL}_2(\mathbb{Z})$, and that the image of M_k in $H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm), \omega^{\otimes k})$ is defined by the condition of "holomorphicity at the cusps".

Thus, we have related modular forms to smooth functions on $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})$. Note that

$$[\mathrm{GL}_2] = \mathrm{GL}_2(\mathbb{Q}) A_{\mathrm{GL}_2} \backslash \mathrm{GL}_2(\mathbb{A}),$$

but our functions φ_f are not invariant under A_{GL_2}, namely

$$\varphi(g, g_\infty r) = r^{-k} \varphi(g, g_\infty)$$

for $r \in A_{\mathrm{GL}_2} \cong \mathbb{R}_{>0}$. However, it is not difficult to rectify this. Let

$$|-|_{\text{adélic}} \colon \mathbb{A}^\times \to \mathbb{R}_{>0},\ ((x_p)_p, x_\infty) \mapsto \prod_p |x_p|_p |x_\infty|$$

be the adélic norm (which is trivial on $\mathbb{Q}^\times$). Then for $f \in M_k$ the function

$$\widetilde{\varphi}_f(g, g_\infty) := |\det(g, g_\infty)|_{\text{adélic}}^{k/2} \varphi_f(g, g_\infty)$$

is invariant under $\mathrm{GL}_2(\mathbb{Q}) A_{\mathrm{GL}_2}$, i.e., defines a smooth function $[\mathrm{GL}_2] \to \mathbb{C}$.

We have to clarify when modular forms give rise to functions in $L^2([\mathrm{GL}_2])$. For this, pick a section $f \in H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}, \omega^{\otimes k})$. The question is: When is

$$\widetilde{\varphi}_f \in C^\infty([\mathrm{GL}_2])$$

in $L^2([\mathrm{GL}_2])$? Pick a compact-open subgroup $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ such that

$$f \in H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^{\pm}), \omega^{\omega k}).$$

The function $\widetilde{\varphi}_f$ is L^2 if and only if the smooth function $\widetilde{\varphi}_f \in C^\infty([\mathrm{GL}_2]/K) \subseteq C^\infty([\mathrm{GL}_2])$ is L^2 (as K is compact). This reduces us to the question: For $\Gamma \subseteq \mathrm{GL}_2(\mathbb{Z})$ some congruence subgroup, $f \in H^0(\Gamma\backslash\mathbb{H}^{\pm}, \omega^{\otimes k})$. When is the function

$$\Gamma A_{\mathrm{GL}_2}\backslash\mathrm{GL}_2(\mathbb{R}) \to \mathbb{C},\ g \mapsto \widetilde{\varphi}_f(g) = |\det(g)|^{k/2}(ci+d)^{-k} f(g(i))$$

square-integrable, i.e., in L^2? We claim:

$$|\varphi_f(g)|^2 = |f(g(i))|^2 |\mathrm{Im}(g(i))|^k$$

for any $g \in \mathrm{GL}_2(\mathbb{R})$, or equivalently,

$$|\det(g)|^{-1/2}|ci+d| \overset{?}{=} |\mathrm{Im}(g(i))|^{1/2}$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$. But this is true as both sides are invariant under

- $K_\infty = \mathrm{SO}_2(\mathbb{R})$-invariant from the right,
- the subgroup $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{R})$ acting from the left,
- the diagonal matrices in $\mathrm{GL}_2(\mathbb{R})$ acting from the left.

Hence

$$\int\limits_{\Gamma A_{\mathrm{GL}_2}\backslash\mathrm{GL}_2(\mathbb{R})} |\varphi_f(g)|^2 dg = \int\limits_{\Gamma\backslash\mathbb{H}^{\pm}} |f(z)|^2 |y|^k \frac{1}{y^2} dx \wedge dy$$

(Recall that we integrate over "the" invariant measure of $\mathbb{H}^{\pm}$.) As $\Gamma\backslash\mathbb{H}^{\pm}$ has finitely many cusps this leads us to a local statement at cusps. By conjugation we may reduce to consider $\infty \in \mathbb{P}^1(\mathbb{Q})$. Then write f in Fourier expansion at ∞, i.e.,

$$f(z) = \sum_{n\in\mathbb{Z}} a_n q^{n/h}.$$

for some $h \geq 0$ (depending on Γ). Then

$$|y|^k f(z) = 2\pi |\log(|q|)|^k \sum_{n\in\mathbb{Z}} a_n q^{n/h}$$

as $y = \log(|q|)$. Now for $0 < r \leq 1$

$$\int\limits_{0<|q|\leq r} |\log(|q|)|^{k-2} \sum_{n\in\mathbb{Z}} a_n q^{n/h} \frac{i}{2} dq \wedge d\overline{q} < \infty$$

if $a_n = 0$ for $n \leq 0$, i.e., if f is cuspidal.

Thus we get, as a summary of the previous discussion, that the map

$$M_k \subseteq H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}), \omega^k) \to C^\infty([\mathrm{GL}_2]),\ f \mapsto \widetilde{\varphi}_f$$

induces a canonical inclusion

$$\widetilde{\Phi}\colon S_k \to L^2([\mathrm{GL}_2]),$$

where $S_k := \varinjlim_K S_k(K)$.

We can make the following observations:

- $\mathrm{GL}_2(\mathbb{A}_f)$ acts on $H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}^{\pm}),\omega^k)$, $C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$, $C^\infty([\mathrm{GL}_2])$, $L^2([\mathrm{GL}_2])$, M_k, S_k
- The morphism
$$\Phi\colon M_k \to C^\infty(\mathrm{GL}_2(\mathbb{A})),\ f \mapsto \varphi_f$$
is $\mathrm{GL}_2(\mathbb{A})$-equivariant, but not
$$M_k \to C^\infty([\mathrm{GL}_2]),\ f \mapsto \widetilde{\varphi}_f = |\det|_{\text{adélic}}^{k/2}\varphi_f$$
(because of the factor $|\det|_{\text{adélic}}^{k/2}$).
- More precisely, for $g \in \mathrm{GL}_2(\mathbb{A}_f)$ and varying $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ the isomorphism
$$M_k(K) \xrightarrow{g} M_k(g^{-1}Kg) \subseteq M_k$$
defines the action $M_k \xrightarrow{g} M_k$ as $M_k = \varinjlim_K M_k(K)$.
- $\mathrm{GL}_2(\mathbb{A}_f)$ does not act on $M_k(K), S_k(K), \ldots$ for any fixed $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$.

The existence of the $\mathrm{GL}_2(\mathbb{A}_f)$-action is the main motivation why we switched from Γ-modular forms to K-modular forms.

We thus see our next tasks:

- Describe S_k as a $\mathrm{GL}_2(\mathbb{A}_f)$-representation.
- Characterize the image of S_k in $L^2([\mathrm{GL}_2])$ (or of M_k in $C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$).

Hopefully, the link of modular forms to automorphic representations (in the L^2-sense) becomes more clear now. For $f \in S_k$ we can take the closure of the subspace generated by the $\mathrm{GL}_2(\mathbb{A})$-translates of f and hope that it defines an irreducible, unitary representation. Note that the representations obtained in this way lie automatically in the discrete part $L^2_{\text{disc}}([\mathrm{GL}_2])$ of $L^2([\mathrm{GL}_2])$.

4. From modular forms to automorphic representations, part II

Today:

- Continue construction of automorphic representations associated to cusp forms.

Last time:

- We constructed a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant embedding
$$\Phi\colon M_k \to C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})),$$
which induces an embedding
$$\widetilde{\Phi}\colon S_k \to L^2([\mathrm{GL}_2]),$$
which is $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant up to multiplying with $|\det|_{\text{adélic}}^{k/2}$, i.e., the morphism
$$\widetilde{\Phi}\colon S_k \otimes |\det|_{\text{adélic}}^{k/2} \to L^2([\mathrm{GL}_2])$$
is $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant.
- Here
$$M_k = \varinjlim_{K \subseteq \mathrm{GL}_2(\mathbb{A}_f)} M_k(K)$$
with
$$M_k(K) = \bigoplus_{i=1}^{m} M_k(\Gamma_i)$$
a sum of spaces of modular forms for congruence subgroups $\Gamma_i \subseteq \mathrm{GL}_2(\mathbb{Z})$.
- Similarly:
$$S_k = \varinjlim_{K} S_k(K)$$
with $S_k(K)$ a sum of spaces of cusp forms for congruence subgroups.
- Today: How can one characterize the images
$$\Phi(M_k) \subseteq C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$$
resp.
$$\widetilde{\Phi}(S_k) \subseteq L^2([\mathrm{GL}_2])?$$
.
- Next time: How does S_k decompose as a $\mathrm{GL}_2(\mathbb{A}_f)$-representation?

The construction of Φ from the last lecture:

- We considered the $\mathrm{GL}_2(\mathbb{R})$-equivariant line bundles
$$L_k \to \mathbb{H}^\pm \cong \mathrm{GL}_2(\mathbb{R})/\mathbb{C}^\times$$
associated to the representation $z \mapsto z^k$ of $\mathbb{C}^\times = A_{\mathrm{GL}_2}\mathrm{SO}_2(\mathbb{R})$.
- By pullback we obtained line bundles $\omega^{\otimes k}$ on $\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm)$.
- Smooth, holomorphic,... sections of $\omega^{\otimes k}$ identify with smooth, holomorphic,... functions
$$f\colon \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm \to \mathbb{C},\ (g,z) \mapsto f(g,z)$$
satisfying
$$f(\gamma g, \gamma z) = j(\gamma, z)^k f(g,z)$$

for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}),$$

and

$$j(\gamma, z) := cz + d.$$

- The function

$$j\colon \mathrm{GL}_2(\mathbb{R}) \times \mathbb{H}^{\pm} \to \mathbb{C}^{\times},\ (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \mapsto cz + d.$$

  satisfies the "cocycle equation"

$$j(gh, z) = j(g, hz)j(h, z)$$

  for all $g, h \in \mathrm{GL}_2(\mathbb{R}), z \in \mathbb{H}^{\pm}$.
- For $f\colon \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm} \to \mathbb{C}$ as above the function

$$\varphi_f\colon \mathrm{GL}_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C},\ (g, g_\infty) \mapsto j(g_\infty, i)^{-k} f(g, g_\infty i)$$

  is $\mathrm{GL}_2(\mathbb{Q})$-equivariant, i.e., lies in $C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$.
- We obtain by Lemma 3.2 that the map $f \mapsto \varphi_f$ identifies *smooth* sections

$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm} \to \omega^{\otimes k}$$

  with smooth functions

$$\varphi\colon \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$$

  satisfying

$$\varphi(g, g_\infty z) = z^{-k}\varphi(g, g_\infty)$$

  for all $(g, g_\infty) \in \mathrm{GL}_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{R})$, $z \in \mathbb{C}^{\times}$.

Upshot and next aims:

- Describe the image of M_k in $C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$ under $f \mapsto \varphi_f$ by conditions on $\varphi \in C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$.
- Imposing the condition of cuspidality will then determine the image of S_k under

$$\widetilde{\Phi}\colon S_k \to L^2([\mathrm{GL}_2])$$

  as those smooth cuspidal functions such that $|\det|_{\text{adélic}}^{-k/2}$ lies in the image of Φ.
- We already know that an element φ in the image of M_k must satisfy

$$\varphi(g, g_\infty z) = z^{-k}\varphi(g, g_\infty)$$

  for $z \in \mathbb{C}^{\times}$ and $(g, g_\infty) \in \mathrm{GL}_2(\mathbb{A})$ and that such φ descent to a smooth function

$$f_\varphi\colon \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm} \to \mathbb{C}, (g, z) \mapsto f(g, z)$$

  satisfying modularity for $\mathrm{GL}_2(\mathbb{Q})$.
- Need to check:
  - When is f_φ holomorphic?
  - When is f_φ holomorphic at the cusps?

When is f_φ holomorphic? This reduces to the following question: Given a smooth function

$$f\colon \mathbb{H}^{\pm} \to \mathbb{C}.$$

Which condition on

$$\varphi_f\colon \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C},\ g \mapsto j(g,i)^{-k} f(gi)$$

guarantees that f is holomorphic? We outline the strategy we will follow:

- The Lie algebra
$$\mathfrak{g} := \mathfrak{gl}_2(\mathbb{R}) \cong \mathrm{Mat}_{2,2}(\mathbb{R})$$
of $\mathrm{GL}_2(\mathbb{R})$ acts on $C^\infty(\mathrm{GL}_2(\mathbb{R}))$ by deriving the action of $\mathrm{GL}_2(\mathbb{R})$ by *right* translations on $\mathrm{GL}_2(\mathbb{R})$.
- Make this action explicit on φ_f.
- Then construct an element $Y \in \mathfrak{g}_{\mathbb{C}}$ such that
$$Y * \varphi_f = 0$$
if and only if f is holomorphic. Here $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is the complexified Lie algebra of $\mathfrak{g}$.

Infinitesimal actions (cf. [GH19, Section 4.2]**):** We start to explain what we meant by "deriving the $\mathrm{GL}_2(\mathbb{R})$" action.

- Let H any Lie group with Lie algebra $\mathfrak{h} := T_1 H$. Here $T_1 H$ denotes the tangent space of the real manifold H at the identity element, i.e., the elements of $\mathfrak{h} = T_1 H$ are equivalences of paths
$$\gamma\colon (-\varepsilon, \varepsilon) \to H$$
with $\varepsilon > 0, \gamma(0) = 1$, and two paths are equivalent if they have the same derivative at 1 (in some chart).
- Let V be any representation of H, where V a Hausdorff topological $\mathbb{C}$-vector space.
- For $X \in \mathfrak{h}$, defined by some path $\gamma\colon (-\varepsilon,\varepsilon) \to H$ with $\gamma(0) = 1$, and $v \in V$ consider the function
$$(-\varepsilon,\varepsilon) \setminus \{0\} \to V,\ t \mapsto \frac{\gamma(t)v - v}{t}.$$
If the limit of this function for $t \to 0$ exists, we write
$$X * v := \varinjlim_{t \to 0} \frac{\gamma(t)v - v}{t}.$$
- An element $v \in V$ is called smooth if
$$X_1 * (\ldots * (X_n * v))\ldots)$$
exists for all $X_1, \ldots, X_n \in \mathfrak{h}$.
- We call $V_{\mathrm{sm}} \subseteq V$ the subspace of smooth vectors in V.
- Then V_{sm} is stable under H (but maybe different from V).
- Moreover, V_{sm} is a representation of $\mathfrak{h}$, or equivalently a module under the enveloping algebra for $\mathfrak{h}$:
$$U(\mathfrak{h}) := \bigoplus_{n \geq 0} \mathfrak{h}^{\otimes n} / \langle X \otimes Y - Y \otimes X - [X,Y] \mid X, Y \in \mathfrak{h} \rangle.$$
- Here: $[-,-]$ the Lie bracket of $\mathfrak{h}$.

The explicit action of $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{R})$ on φ_f: We now make the action of $\mathfrak{gl}_2(\mathbb{R})$ on some function

$$\varphi_f(g) := j(g,i)^{-k} f(gi)$$

associated to some section

$$f\colon \mathbb{H}^\pm \to \mathcal{O}_{\mathbb{H}^\pm}(k)$$

of $\mathcal{O}_{\mathbb{H}^\pm}(k)$ more explicit. For this let us recall the equation

$$j(gh,z) = j(g,hz)j(h,z)$$

for $g, h \in \mathrm{GL}_2(\mathbb{R}), z \in \mathbb{H}^\pm$.

- We already know

  $$\varphi_f(gz) = z^{-k}\varphi_f(g)$$

  for $z \in \mathbb{C}^\times$, $g \in \mathrm{GL}_2(\mathbb{R})$ (by Lemma 3.2). Note that

  $$\mathbb{C} \cong \mathrm{Lie}(\mathbb{C}^\times) = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle_{\mathbb{R}} \subseteq \mathfrak{g}.$$

  By deriving we can conclude that $\mathbb{C}$ acts on φ_f via the linear form

  $$\mathbb{C} \mapsto \mathbb{C},\ z \mapsto -kz.$$

  Thus, more precisely

  $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \varphi_f = -k\varphi_f$$

  and

  $$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} * \varphi_f = -ki\varphi_f,$$

  because $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ corresponds to $i \in \mathbb{C}$.
- Consider now the subgroup

  $$U := \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{R})$$

  with associated Lie algebra

  $$\mathrm{Lie}(U) = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle_{\mathbb{R}} \subseteq \mathfrak{g}.$$
- Define the auxiliary function

  $$\tilde{\varphi}_f\colon \mathrm{GL}_2(\mathbb{R}) \times \mathbb{H}^\pm \to \mathbb{C},\ (g,z) \mapsto j(g,z)^{-k} f(gz).$$

  (do not confuse it with $\widetilde{\varphi}_f$ which was introduced in the last lecture).
- For a fixed $g \in \mathrm{GL}_2(\mathbb{R})$ we obtain

  $$\frac{\partial \tilde{\varphi}_f}{\partial \overline{z}}(g,z) = \det(g) j(g,z)^{-k-2} \frac{\partial f}{\partial \overline{z}}(gz)$$

  (using that the function $z \mapsto gz$ has $\frac{\partial}{\partial z}$-derivative $\frac{\det(g)}{j(g,z)^2}$.)
- Thus, f is holomorphic if and only if

  $$\frac{\partial \tilde{\varphi}_f}{\partial \overline{z}}(g,i) = 0$$

  for all $g \in \mathrm{GL}_2(\mathbb{R})$.

- For any path $\gamma\colon (-\varepsilon,\varepsilon)\to U$ with $\gamma(0)=1$ we can calculate

$$\begin{array}{rc} & \varphi_f(g\gamma(t)) \\ = & \tilde{\varphi}_f(g\gamma(t),i) \\ = & j(g\gamma(t),i)^{-k}f(g\gamma(t)i) \\ = & j(g,\gamma(t)i)^{-k}j(\gamma(t),i)^{-k}f(g\gamma(t)i) \\ = & \tilde{\varphi}_f(g,\gamma(t)i) \end{array}$$

  because $j(\gamma(t),i)=1$.
- Set

$$X_\gamma := \frac{\partial\gamma}{\partial t}(0)\in \mathrm{Lie}(U).$$

- Then

$$\begin{array}{rl} & X_\gamma * \varphi_f \\ = & \frac{\partial}{\partial t}(\varphi_f(g\gamma(t))_{|t=0} \\ = & \frac{\partial\tilde{\varphi}_f}{\partial z}(g,i)\frac{\partial}{\partial t}(\gamma(t)\cdot i)_{|t=0} + \frac{\partial\tilde{\varphi}_f}{\partial \bar{z}}(g,i)\overline{\frac{\partial}{\partial t}(\gamma(t)\cdot i)}_{|t=0}. \end{array}$$

  using the chain rule for Wirtinger derivatives.
- More concretely, consider as a first path the function

$$\gamma_1\colon (-\varepsilon,\varepsilon)\to U,\ t\mapsto \begin{pmatrix} 1+t & 0 \\ 0 & 1\end{pmatrix}.$$

  Then $\gamma_1(t)\cdot i=(1+t)i$ and thus

$$\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} * \varphi_f(g) = i\frac{\partial\tilde{\varphi}_f}{\partial z}(g,i) - i\frac{\partial\tilde{\varphi}_f}{\partial \bar{z}}(g,i).$$

- As a second path we take

$$\gamma_2\colon (-\varepsilon,\varepsilon)\to U,\ t\mapsto \begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix},$$

  then $\gamma_1(t)\cdot i = i+t$ and thus

$$\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} * \varphi_f(g) = \frac{\partial\tilde{\varphi}_f}{\partial z}(g,i) + \frac{\partial\tilde{\varphi}_f}{\partial \bar{z}}(g,i).$$

- In particular,

$$i\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} * \varphi_f(g) + \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} * \varphi_f(g) = 2\frac{\partial\tilde{\varphi}_f}{\partial \bar{z}}(g,i)$$

- Thus

$$\begin{pmatrix} i & 1 \\ 0 & 0\end{pmatrix} \in \mathfrak{g}_{\mathbb{C}}.$$

  would be a candidate for Y, i.e., f is holomorphic if and only if $\begin{pmatrix} i & 1 \\ 0 & 0\end{pmatrix} * \varphi_f = 0$.
- But we can do better. Namely, define

$$\begin{array}{rl} Y := & \frac{1}{2}\begin{pmatrix} -1 & i \\ i & 1\end{pmatrix} \\ & = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} + \frac{1}{2}i\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} + i\begin{pmatrix} i & 1 \\ 0 & 0\end{pmatrix} \end{array}.$$

Then

$$Y * \varphi_f(g) = \underbrace{-\frac{k}{2}\varphi_f(g) + i(-\frac{ki}{2})\varphi_f(g)}_{=0} + 2i\frac{\partial\tilde{\varphi}_f}{\partial\overline{z}}(g,i) = 2i\frac{\partial\tilde{\varphi}_f}{\partial\overline{z}}(g,i),$$

and thus f is holomorphic if and only if $Y * \varphi_f = 0$.

- Why is this choice better?
  - Set
$$H := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X := \frac{1}{2}\begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix}.$$
  - Then H, X, Y is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_{\mathbb{C}}$.
  - Namely, in the basis
$$\begin{pmatrix} -i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix},$$
    it is given by
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$
  - Note that H spans the Lie algebra $i\mathrm{Lie}(\mathrm{SO}_2(\mathbb{R})) \subseteq \mathfrak{g}_{\mathbb{C}}$.

We can pass to the next question.

When is f_φ holomorphic at the cusps?

- Consider a function in Fourier expansion
$$f(z) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i z n},$$
  defined on
$$\{z \in \mathbb{C} \mid |\mathrm{Re}(z)| < 1, \mathrm{Im}(z) > 1\}.$$
- Recall that
$$|e^{2\pi i z}| = e^{-2\pi y}$$
  with $y = \mathrm{Im}(z)$.
- Then $a_n = 0$ for all $n < 0$ if and only if
$$|f(x+iy)| \leq Cy^N$$
  for some $C, N \in \mathbb{R}_{>0}$ and $y \to \infty$, because for $n < 0$ the exponential function $e^{-2\pi n y}$ grows faster than any polynomial function.
- By definition this means f is of *moderate growth*, or *slowly increasing*, a notion which can be generalized to all reductive group over $\mathrm{Spec}(\mathbb{Q})$.
- Let us note that moreover, $a_n = 0$ for all $n \leq 0$, i.e., including $n = 0$, if and only if for all $N \in \mathbb{R}_{>0}$
$$|f(x+iy)| \leq Cy^{-N}$$
  for some $C \in \mathbb{R}_{>0}$ and $y \to \infty$.
- In other words, f is *rapidly decreasing*.

Moderate growth for general reductive groups: The moderate growth condition can be phrased for arbitrary reductive groups G over $\mathbb{Q}$ as follows

- For each embedding $\rho\colon G \hookrightarrow \mathrm{SL}_{n,\mathbb{Q}}$ (not into $\mathrm{GL}_{n,\mathbb{Q}}$) we obtain a norm
$$|g| := \sup_v \max_{i,j=1,\ldots,n} \{|\rho(g)_{i,j}{}_v|_v\}$$
  on $G(\mathbb{A})$, where v runs through all primes and ∞.

- A function $\varphi\colon G(\mathbb{A}) \to \mathbb{C}$ is of moderate growth or slowly increasing if
$$|\varphi(g)| \leq C|g|^N$$
for suitable $C, N \in \mathbb{R}_{>0}$, cf. [GH19, Definition 6.4.].
- It is not difficult to see that this definition does not depend on the embedding ρ (as this changes the norm only up to some polynomial functions).

We leave it as an exercise (cf. [GH19, Lemma 6.3.1.] and our above remark on Fourier coefficients) to check that a (holomorphic) section
$$f \in H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}), \omega^{\otimes k}).$$
is holomorphic at the cusps if and only of the function $\varphi_f \in C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$ is of moderate growth.

We give a short summary of what we have done.

Upshot:

- We can describe the image of
$$M_k \to C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})),\ f \mapsto \varphi_f.$$
- Namely, $\varphi \in C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$ lies in the image if and only if
  - $\varphi(g, g_\infty z) = z^{-k}\varphi(g, g_\infty z)$ for $(g, g_\infty) \in \mathrm{GL}_2(\mathbb{A})$, $z \in \mathbb{C}^\times$.
  - φ is of moderate growth.
  - For each $g \in \mathrm{GL}_2(\mathbb{A}_f)$
  $$Y * \varphi(g, -) = 0,$$
  where $Y \in \mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_2(\mathbb{R})_{\mathbb{C}}$ is the element constructed before.
- Note that by our discussion these conditions match the conditions "modularity, holomorphic at cusps, holomorphic" in the definition of a modular form.
- For $k \geq 1$ it is not difficult to see that there exists a unique, up to isomorphism, irreducible $(\mathfrak{g}_{\mathbb{C}}, \mathrm{O}_2(\mathbb{R}))$-module D'_{k-1} containing a non-zero element v, such that $Y * v = 0$, and $\mathbb{C}^\times$ acts on v via the character $z \mapsto z^{-k}$, cf. [Del73, Section 2.1] (where the dual of D'_{k-1} is described).
- Using D'_{k-1} one can rewrite our result as saying that
$$M_k \cong \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}}, \mathrm{O}_2(\mathbb{R}))}(D'_{k-1}, C^\infty_{\mathrm{mg}}(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))),$$
where $C^\infty_{\mathrm{mg}}(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})) \subseteq C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$ denotes the subset of functions with moderate growth.
- More on this will appear later when we discuss automorphic forms and $(\mathfrak{g}_{\mathbb{C}}, K_\infty)$-modules, cf. Section 11.

We finish the lecture by sketching how to incorporate cuspidality.

A condition for cuspidality:

- Let $\varphi = \varphi_f \in C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$ for some $f \in M_k$, seen as a function on $\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm} \to \mathbb{C}$ satisfying modularity.
- Then $f \in S_k$ if and only if the integral
$$\varphi_B(g) := \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)dn$$

vanishes for all $g \in \mathrm{GL}_2(\mathbb{A})$, where

$$N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

is the unipotent radical in the standard Borel

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

and dn an $N(\mathbb{A})$-invariant measure on $N(\mathbb{Q})\backslash N(\mathbb{A})$.

- Namely:
  - We may assume that f, and thus $\varphi = \varphi_f$, is invariant under some principal congruence subgroup $K(m) \subseteq \mathrm{GL}_2(\mathbb{A}_f)$.
  - Consider $g = 1$. Then using $N(\mathbb{Q})\backslash N(\mathbb{A}) \cong (m\widehat{\mathbb{Z}} \times \mathbb{R})/m\mathbb{Z}$:

$$\begin{aligned} & \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi_f(n)dn \\ = & \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} j(n_\infty, i)^{-k} f(n, n_\infty i) d(n, n_\infty) \\ = & \int\limits_{\mathbb{R}/m\mathbb{Z}} \int\limits_{m\widehat{\mathbb{Z}}} f(a, i+x) da dx \\ = & \mu(m\widehat{\mathbb{Z}}) \int\limits_{\mathbb{R}/m\mathbb{Z}} f(1, i+x)dx \\ = & \mu(m\widehat{\mathbb{Z}}) a_0 \end{aligned}$$

    with $\mu(m\widehat{\mathbb{Z}})$ the volume of $m\widehat{\mathbb{Z}}$ and

$$f(1, z) = \sum_{n=0}^{\infty} a_n q^{n/m}$$

    the Fourier expansion of the modular form $f(1, -)$ for $\Gamma(m)$ at the cusp ∞.
  - Indeed: The crucial point is that

$$\int\limits_{\mathbb{R}/m\mathbb{Z}} e^{2\pi i \frac{n}{m} x} dx = 0$$

    if $n > 0$ (which is part of Fourier theory, and follows more generally from the statement that $\int\limits_A \chi(g)dg = 0$ for any compact, abelian group A and any *non-trivial* character $\chi \colon A \to \mathbb{C}^\times$).
  - The condition $\varphi_B(g) = 0$ for *all* $g \in \mathrm{GL}_2(\mathbb{A})$ is then equivalent to the vanishing of the constant Fourier coefficients at *all* cusps.
  - We leave the details as an exercise, cf. [Del73, Rappel 1.3.2.], and [Gel75, Proposition 3.1.(vii)].
- Note: Because φ is $\mathrm{GL}_2(\mathbb{Q})$-invariant for left translations, we can replace B by any conjugate gBg^{-1} with $g \in \mathrm{GL}_2(\mathbb{Q})$.

Let us note that the way we associate functions on $\mathrm{GL}_2(\mathbb{A})$ is different in formulation than in the standard references, e.g., [Gel75], which usually restrict to modular forms on

$$\Gamma_1(m) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0,\ d \equiv 1 \bmod m\}$$

or to modular forms with nebentype $\chi\colon (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$ for

$$\Gamma_0(m) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod m\}.$$

Although these capture all automorphic representations generated by modular forms, we think it is a bit unmotivated to restrict to these congruence subgroups (or their counterparts $K_1(m), K_0(m) \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ in this moment). We will clarify the relation of our approach with the classical ones in the next lecture.

5. From modular forms to automorphic representations, part III

Last time:

- Let $\varphi \in C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$.
- Then $\varphi = \varphi_f$ for some $f \in M_k$ if and only if
  - $\varphi(g, g_\infty z) = z^{-k}\varphi(g, g_\infty)$ for $(g, g_\infty) \in \mathrm{GL}_2(\mathbb{A})$, $z \in \mathbb{C}^\times$.
  - φ is of moderate growth.
  - For each $g \in \mathrm{GL}_2(\mathbb{A}_f)$
  $$Y * \varphi(g, -) = 0,$$
  where $Y \in \mathfrak{g}_\mathbb{C} = \mathfrak{gl}_2(\mathbb{R})_\mathbb{C}$ is a suitably constructed, natural element.
- $f \in M_k$ is a cusp form if and only if for all $B \subseteq \mathrm{GL}_{2,\mathbb{Q}}$ proper parabolic, i.e., Borel, with unipotent radical $N \subseteq \mathrm{GL}_{2,\mathbb{Q}}$:
  $$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi_f(ng)dn = 0$$
  for all $g \in \mathrm{GL}_2(\mathbb{A})$.

Today:

- Describe S_k as a $\mathrm{GL}_2(\mathbb{A}_f)$-representation.
- This will yield our main source of examples for automorphic representations.[9]

Considering the whole spaces M_k, S_k (and not merely $M_k(K), S_k(K)$ for some compact-open subgroup $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$, or even $M_k(\Gamma)$, $S_k(\Gamma)$ for $\Gamma \in \mathrm{SL}_2(\mathbb{Z})$ a congruence subgroup) has the advantage that one now can use representation theory for the locally profinite group $\mathrm{GL}_2(\mathbb{A}_f)$. In this vein the following observation is important.

Observation:

- M_k and S_k are *smooth* and *admissible* $\mathrm{GL}_2(\mathbb{A}_f)$-representations in the following sense.

Definition 5.1. *Let G be a locally profinite group (like $\mathrm{GL}_2(\mathbb{A}_f)$). A representation G on a $\mathbb{C}$-vector space V is called smooth if*
$$V = \bigcup_{K \subseteq G} V^K,$$
where K runs through the compact-open subgroups of G and
$$V^K := \{v \in V \mid kv = v \text{ for all } k \in K\}.$$
Equivalently, V is smooth if the action morphism
$$G \times V \to V$$
is continuous with V carrying the discrete topology. Denote by
$$\mathrm{Rep}^\infty_\mathbb{C} G$$
the (abelian) category of smooth representations of G on $\mathbb{C}$-vector spaces.

[9] More precisely, the irreducible subrepresentations of the closure of $\widetilde{\Phi}(S_k) \subseteq L^2([\mathrm{GL}_2])$ are the automorphic representations (in the L^2-sense) associated with cusp forms.

Definition 5.2. *Let G be a locally profinite group. A smooth representation V of G is called admissible if*

$$\dim_{\mathbb{C}} V^K < \infty$$

for all $K \subseteq G$ compact-open.

Let us consider some examples

- $M_k = \bigcup\limits_{K \subseteq \mathrm{GL}_2(\mathbb{A}_f)} M_k^K$ with $M_k^K = M_k(K)$ is smooth and admissible because the $M_k(K)$ are finite dimensional.
- Similarly: S_k is an admissible representation of $\mathrm{GL}_2(\mathbb{A}_f)$.
- For G reductive over $\mathbb{Q} \Rightarrow C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is a smooth $G(\mathbb{A}_f)$-representation, but not admissible.
- $L^2([G])$ is not a smooth $G(\mathbb{A}_f)$-representation.

In Section 11 we will introduce the space of automorphic forms for a reductive group G over $\mathbb{Q}$ which serves as a replacement (actually a huge generalization) for the space M_k, which is again a smooth $G(\mathbb{A}_f)$-representation. As we will see smooth representations are more algebraic in nature than unitary representations, like $L^2([G])$. This makes smooth representations of locally profinite groups more convenient to work with.

In the classical theory of modular forms Hecke operators (which might seem to be some ad hoc definition) are important. From the representation-theoretic perspective they arise from the action of an *Hecke algebra*.

The Hecke algebra (of a locally profinite group):

- G a locally profinite group.
  - e.g., $\mathrm{GL}_2(\mathbb{A}_f)$, $\mathrm{GL}_n(\mathbb{Q}_p)$, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \ldots$
- For $K \subseteq G$ compact-open, there exists a unique left-invariant resp. a unique right-invariant Haar measure μ on G with $\mu(K) = 1$.
- Concretely:
  - By translation invariance it is sufficient to define μ for $K' \subseteq G$ a compact-open subgroup.
  - Define $K'' := K' \cap K$ compact-open. Then K'' is of finite index in K resp. K'.
  - Set
    $$\mu(K') := \frac{[K' : K'']}{[K : K'']}.$$
  - This defines the desired measure and is also the unique possible choice.
  - Note that μ takes actually values in $\mathbb{Q}$. This is a first hint that the theory of smooth representations of locally profinite groups is "more algebraic".
- From now on, fix a left invariant Haar measure μ on G.
- Define the associated "Hecke algebra"
  $$\mathcal{H}(G) := C_c^\infty(G)$$
  of G as the set of locally constant, compactly supported functions $G \to \mathbb{C}$.
- $\mathcal{H}(G)$ is an algebra via convolution:
  $$f * g(x) := \int\limits_G f(xy)g(y^{-1})dy$$

for $f, g \in \mathcal{H}(G)$. Note that $\mathcal{H}(G)$ depends implicitly on the chosen measure. However, it is possible to define $\mathcal{H}(G)$ intrinsically as the space of compactly supported distributions with multiplication given by convolution of distributions, cf. [BRa, Definition 7].

- The multiplication in $\mathcal{H}(G)$ is a purely algebraic operation:
  - Each $f \in \mathcal{H}(G)$ is a finite sum
  $$f = \sum_{i=1}^{m} a_i \chi_{g_i K_i}$$
  for some $g_i \in G$, $K_i \subseteq G$ compact-open subgroups.
  - Wlog: $K_1 = \ldots = K_m$ (by shrinking).
  - Let $K \subseteq G$ be compact-open and $g_1, g_2 \in G$. Then
  $$g_1 K g_2 K = \coprod_{j=1}^{m} h_j K$$
  for some $h_j \in G$ and
  $$\chi_{g_1 K} * \chi_{g_2 K} = \sum_{j=1}^{m} \mu(K) \chi_{h_j K}.$$
- The algebra $\mathcal{H}(G)$ has a lot of idempotents, namely,
$$e_K := \frac{1}{\mu(K)} \chi_K$$
with $K \subseteq G$ compact-open subgroups, χ_K the characteristic function of K. But in general $\mathcal{H}(G)$ has no identity (unless G is compact).

Modules for the Hecke algebra (cf. [BRa]**):**

- $\mathcal{H}(G)$ is a replacement for the group algebra $\mathbb{C}[G]$, which is better suited to study *smooth* representations of G.
- If $V \in \mathrm{Rep}_{\mathbb{C}}^{\infty} G$, then
$$f * v := \int_G f(h) h v dh$$
for $f \in \mathcal{H}(G)$, $v \in V$, defines an action of $\mathcal{H}(G)$ on V.
- Concretely: If $K \subseteq G$ is a compact-open subgroup and sufficiently small, such that $v \in V$ is fixed by K and $f = \sum_{i=1}^{m} a_i \chi_{g_i K}$ with $a_i \in \mathbb{C}, g_i \in G$, then
$$f * v = \sum_{i=1}^{m} a_i g_i v.$$
- Assume $K \subseteq G$ is a compact-open subgroup. We define the Hecke algebra of G relative to K
$$\mathcal{H}(G, K) := e_K \mathcal{H}(G) e_K.$$
Then $\mathcal{H}(G, K)$ is an algebra with unit e_K.
- Let V be any smooth representation of G. Then it is not difficult to see that
$$V^K = e_K * V$$
for any $K \subseteq G$ compact-open.

- Applying this to the smooth G-representation $\mathcal{H}(G)$ (via left and right multiplication) we obtain that
$$\mathcal{H}(G,K) \cong C_c^\infty(K\backslash G/K),$$
i.e., $\mathcal{H}(G)$ identifies with the set of K-biinvariant, compactly supported functions on G.

The following small statement is important in our upcoming discussion of the $\mathrm{GL}_2(\mathbb{A}_f)$-representation S_k.

- If $V \in \mathrm{Rep}_{\mathbb{C}}^\infty G$ is irreducible and $V^K \neq 0$, then V^K is irreducible as an $\mathcal{H}(G,K)$-module.
  - Indeed: If $M \subseteq V^K$ is a non-trivial $\mathcal{H}(G,K)$-submodule, then
$$V = \mathcal{H}(G) * M$$
by irreducibility V and thus
$$V^K = e_K(\mathcal{H}(G) * M)) = e_K\mathcal{H}(G)e_K M = M,$$
because $e_K * M = M$.
- Conversely one can show that if V^K is irreducible or zero for all compact-open subgroups $K \subseteq G$, then V is irreducible (if $V \neq 0$).

Namely, this property allows as to associate *Hecke eigensystems* to irreducible, admissible $\mathrm{GL}_2(\mathbb{A}_f)$-representations.

From irreducible, admissible $\mathrm{GL}_n(\mathbb{A}_f)$-representations to Hecke eigensystems:

- Let V be an irreducible, admissible $\mathrm{GL}_n(\mathbb{A}_f)$-representation.
- Recall that the compact-open subgroups
$$K(m) = \ker(\mathrm{GL}_n(\widehat{\mathbb{Z}}) \to \mathrm{GL}_n(\mathbb{Z}/m)),\ m \geq 0,$$
form basis of compact-open neighborhoods of the identity in $\mathrm{GL}_2(\mathbb{A}_f)$.
- From the important property and admissibility (which easily implies a version of Schur's lemma) we know that
  - $V^{K(m)}$ is an irreducible $\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_f), K(m))$-module for $m \gg 0$.
  - $\mathrm{End}_{\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_f),K(m))}(V^{K(m)}) \cong \mathbb{C}$.
- In the following we will use that for each prime p a smooth $\mathrm{GL}_2(\mathbb{A}_f)$-representation yields by restricting along the inclusion $\mathrm{GL}_2(\mathbb{Q}_p) \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ a smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation.
- Set
$$S := \bigcap_{m\geq 0,\ V^{K(m)}\neq 0} \{p \text{ prime dividing } m\},$$
- Note that $p \in S$ if and only if $V^{\mathrm{GL}_2(\mathbb{Z}_p)} = 0$.
- We call a prime p unramified for V if $p \notin S$, i.e., if $V^{\mathrm{GL}_2(\mathbb{Z}_p)} \neq 0$.
- The following observation is very important:

  For each prime p the algebra $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$ is commutative!
- Even more is true: Take $m \geq 1$ and $p \nmid m$. Then $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$ is central in $\mathcal{H}(\mathrm{GL}_2(\mathbb{A}_f), K(m))$. More precisely, to a double coset of some $A \in \mathrm{GL}_2(\mathbb{Q}_p)$ one can associate the double coset of the element $(1,\ldots,1,A,1,\ldots,1) \in \mathrm{GL}_2(\mathbb{A}_f)$ whose entry at p is A, and the identity elsewhere, and the resulting embedding
$$\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p)) \to \mathcal{H}(\mathrm{GL}_2(\mathbb{A}_f), K(m))$$

has image in the center of $\mathcal{H}(\mathrm{GL}_2(\mathbb{A}_f), K(m))$. Namely, it is easy to see that the double coset given by

$$(1, \dots, 1, A, 1, \dots, 1)$$

with $A \in \mathrm{GL}_2(\mathbb{Q}_p)$ commutes with double cosets of $B \in \mathrm{GL}_2(\mathbb{A}_f)$ if the component of B at p is the identity matrix. Thus, centrality of $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$ follows actuality from commutativity of $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$.

- To prove the commutativity we use Gelfand's trick. Namely, consider the antiinvolution

$$\sigma\colon \mathrm{GL}_n(\mathbb{Q}_p) \to \mathrm{GL}_n(\mathbb{Q}_p),\ g \mapsto g^{\mathrm{tr}},$$

which has the following properties:
  - σ preserves $\mathrm{GL}_n(\widehat{\mathbb{Z}}^S)$.
  - $\sigma(f_1 * f_2) = \sigma(f_2) * \sigma(f_1)$ for $f_1, f_2 \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$.
  - The cosets $\mathrm{GL}_n(\mathbb{Z}_p) g \mathrm{GL}_n(\mathbb{Z}_p)$ with $g \in \mathrm{GL}_n(\mathbb{Q}_p)$ a diagonal matrix span $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$ (by the elementary divisor theorem).
  - Now we can prove commutativity. Namely, $\sigma(f) = f$ for all $f \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$ and thus

$$f_1 * f_2 = \sigma(f_1 * f_2) = \sigma(f_2) * \sigma(f_1) = f_2 * f_1$$

  as desired.
- The centrality of $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$ is important, as it allows to define the system of Hecke eigenvalues of V (which we recall is an irreducible, admissible $\mathrm{GL}_2(\mathbb{A}_f)$-representation).
- Recall that S denotes the set of ramified primes of V, i.e., those primes p such that $V^{\mathrm{GL}_2(\mathbb{Z}_p)} = 0$.
- For $p \notin S$ we will now construct a morphism

$$\sigma_{V,p}\colon \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p)) \to \mathbb{C}$$

of $\mathbb{C}$-algebras, which is naturally assocatiated with V.
- For this pick $p \notin S$, and choose $v \in V$ fixed by $\mathrm{GL}_2(\mathbb{Z}_p)$. Then v is fixed by $K(m)$ for some $m \geq 0$ with $p \nmid m$ (by smoothness of the $\mathrm{GL}_2(\mathbb{A}_f)$-action).
- As we discussed before irreducibility of V implies that $V^{K(m)}$ is irreducible, and thus

$$\mathrm{End}_{\mathcal{H}(\mathrm{GL}_2(\mathbb{A}_f), K(m))}(V^{K(m)}) \cong \mathbb{C}$$

by Schur's lemma.
- As $\mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p))$ is central in $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$ its acts via a homomorphisms of $\mathcal{H}(\mathrm{GL}_2(\mathbb{A}_f), K(m))$-modules on $V^{K(m)}$.
- Thus we obtain our morphism

$$\sigma_{V,p}\colon \mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p)) \to \mathrm{End}_{\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_f), K(m)}(V^{K(m)}) \cong \mathbb{C},$$

which a priori dependents on v and m. But it is not difficult to see $\sigma_{V,p}$ is invariant under enlarging m. This implies that $\sigma_{V,p}$ is actually independent of v and m, and hence canonically associated with V and p.

Definition 5.3. *The system*

$$\{\sigma_{V,p}\colon \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p) \to \mathbb{C}\}_{p \notin S}$$

is called the system of Hecke eigenvalues for V.

We present a slightly different perspective to systems of Hecke eigenvalues. Namely, let S be some finite set of primes, and define the "Hecke algebra away from S" as

$$\mathbb{T}^S := \mathcal{H}(\mathrm{GL}_2(\mathbb{A}_f^S), \mathrm{GL}_2(\widehat{\mathbb{Z}}^S)),$$

where

$$\widehat{\mathbb{Z}}^S := \prod_{p \notin S} \mathbb{Z}_p$$

and

$$\mathbb{A}_f^S := \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^S.$$

Using Gelfand's trick one can see that $\mathbb{T}^S$ is commutative. Moreover, if one developes the appropriate definition (cf. [Fla79]) the "restricted tensor product"

$$\mathbb{T}^S \cong \bigotimes_{p \notin S}' \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p)).$$

Now, we can also make the following definition.

Definition 5.4. *A Hecke eigensystem, or system of Hecke eigenvalues, is a maximal ideal in $\mathbb{T}^S$ where S is a finite set of primes and $\mathbb{T}^S := \mathcal{H}(\mathrm{GL}_n(\mathbb{A}_f^S), \mathrm{GL}_n(\widehat{\mathbb{Z}}^S))$.*

We can collect the following observations.

- $\mathbb{T}^S$ is of countable dimension and thus each Hecke eigensystem has residue field $\mathbb{C}$.
- A Hecke eigensystem for S is equivalently a collection of $\mathbb{C}$-algebra homomorphisms
$$\sigma_p \colon \mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p)) \to \mathbb{C}$$
for $p \notin S$ as before.
- If $S \subseteq S'$ are two finite sets of primes, each Hecke eigensystem for S induces one for S'. We call two Hecke eigensystems
$$\{\sigma_p\}_{p \notin S} \text{ and } \{\sigma'_p\}_{p \notin S'}$$
equivalent if they agree outside some finite set of primes which contains S and S'.
- As we saw before each irreducible, admissible $\mathrm{GL}_n(\mathbb{A}_f)$-representation yields by the above construction a system of Hecke eigenvalues.

We will see that in the Langlands program the system of Hecke eigenvalues associated to modular forms is expected to being arithmetically interesting.

To gain a more useful perspective on Hecke eigensystems, we need a full description of $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$.

A full description of $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$:

- For moment set $G := \mathrm{GL}_2(\mathbb{Q}_p)$ and $K = \mathrm{GL}_2(\mathbb{Z}_p)$.
- We normalize the Haar measure on G such that $\mu(K) = 1$.
- By the elementary divisor theorem the $\mathbb{C}$-vector space $\mathcal{H}(G, K)$ is free on the elements
$$K \begin{pmatrix} p^i & 0 \\ 0 & p^j \end{pmatrix} K$$
with $(i, j) \in \mathbb{Z}^{2,+} := \{(i, j) \in \mathbb{Z}^2 \mid i \geq j\}$.

- We claim that
$$f\colon \mathbb{C}[X_1, X_2^{\pm 1}] \overset{\cong}{\to} \mathcal{H}(G,K)$$
via the morphism
$$\begin{aligned} X_1 &\mapsto K\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}K, \\ X_2 &\mapsto K\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}K. \end{aligned}$$
Here (and in the following) we identify a double coset with its characteristic function.
- Let us verify the claim (without invoking Gelfand's trick):
  - The element
  $$K\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}K$$
  is central in $\mathcal{H}(G,K)$ as
  $$KzK \cdot KgK = KgK \cdot KzK$$
  for all $g \in G$, and $z \in G$ in the (group-theoretic) center.
  - This implies that f is well-defined (note that $\mathbb{C}[X_1, X_2^{\pm 1}]$ is commutative, but $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$ a priori not.
  - Moreover, let us check that f is surjective. By multiplying with $f(X_2)$ it suffices to see that each
  $$K\begin{pmatrix} p^i & 0 \\ 0 & 1 \end{pmatrix}K$$
  with $i \geq 1$ lies in the image of f.
  - Using induction one sees that for $n \geq 1$ the n-fold product
  $$K\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}K \cdot \ldots \cdot K\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}K$$
  agrees with the set of matrices $\mathrm{Mat}_{2,2}(\mathbb{Z}_p)^{\mathrm{val}=n}$ in $\mathrm{Mat}_{2,2}(\mathbb{Z}_p)$ of determinant of valuation n.
  - From the definition of the product in $\mathcal{H}(G,K)$ one concludes that the product
  $$K\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}K \cdot \ldots \cdot K\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}K$$
  is the characteristic function χ of $\mathrm{Mat}_{2,2}(\mathbb{Z}_p)^{\mathrm{val}=n}$.
  - As this characterstic function χ has the characteristic function of
  $$K\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}K$$
  as a summand with coefficient 1 we can conclude (using a small induction) that it lies in the image of f, because each other double coset occuring in $\mathrm{Mat}_{2,2}(\mathbb{Z}_p)^{\mathrm{val}=0}$ has a representative a matrix of the form
  $$\begin{pmatrix} p^j & 0 \\ 0 & p^i \end{pmatrix}$$
  with $i, j \geq 1$ and $i + j = n$.
  - In particular, we see that $\mathcal{H}(G,K)$ must be commutative without using Gelfand's trick.

– Now it is not difficult to conclude that f must be an isomorphism. Indeed,

$$\mathrm{Spec}(\mathcal{H}(G,K)) \subseteq \mathrm{Spec}(\mathbb{C}[X_1, X_2^{\pm 1}])$$

defines a subscheme whose projection to $\mathrm{Spec}(\mathbb{C}[X_2^{\pm 1}])$ is free of infinite rank, but this is by only possible for $\mathrm{Spec}(\mathbb{C}[X_1, X_2^{\pm 1}])$ itself (using integrality of the latter).

We summarize our discussion on systems of Hecke eigenvalues.

Upshot:

- Each irreducible, admissible $\mathrm{GL}_2(\mathbb{A}_f)$-representation V yields canonically a system of Hecke eigenvalues

$$\tilde{a}_p \in \mathbb{C},\ \tilde{b}_p \in \mathbb{C}^\times$$

for all primes p outside the finite set S of primes, which are ramified for V.
- Concretely: If $v \in \mathrm{GL}_2(\mathbb{A}_f)$ is fixed by $K(m)$ and $p \nmid m$, then

$$\tilde{T}_p(v) := \mathrm{GL}_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p) * v = \tilde{a}_p v,$$

where $\widetilde{T}_p$ is "the Hecke operator at p", and

$$\tilde{S}_p(v) := \mathrm{GL}_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p) * v = \tilde{b}_p v.$$

- Note that the system of Hecke eigenvalues is completely determined by the collection of complex numbers

$$\{\tilde{a}_p, \tilde{b}_p\}_{p \notin S}.$$

- Let $Z \subseteq \mathrm{GL}_2$ be the center, i.e., the subgroup of scalar matrices.
- Let $\psi \colon \mathbb{A}_f^\times \cong Z(\mathbb{A}_f) \to \mathbb{C}^\times$ be the central character of V (which exists by Schur's lemma). Then

$$\tilde{b}_p = \psi((1, \ldots, 1, p, 1, \ldots, 1)$$

for p prime such that $\mathbb{Z}_p^\times \subseteq \mathbb{A}_f^\times$ acts trivially on V.
- The central character ψ yields another invariant of the irreducible, admissible $\mathrm{GL}_2(\mathbb{A}_f)$-representation.
- Note that ψ is in general not determined by the systems of Hecke eigenvalues, because finitely many p are missing and no triviality on $\mathbb{Q}^\times$, i.e., "automorphicity", is required.

We now use our discussion to obtain a decomposition of the smooth, admissible $\mathrm{GL}_2(\mathbb{A}_f)$-representation S_k. First, we collect some general observations before we obtain the precise decomposition.

General results on the $\mathrm{GL}_2(\mathbb{A}_f)$-representation S_k:

- The $\mathrm{GL}_2(\mathbb{A}_f)$-representation S_k decomposes into a direct sum

$$S_k \cong \bigoplus_{i \in I} V_i$$

of irreducible, admissible representations V_i.
- Indeed:

  - Up to twisting by the character $|\det|^{k/2}_{\text{adélic}|\mathrm{GL}_2(\mathbb{A}_f)}$ we have a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant embedding $S_k \subseteq L^2([\mathrm{GL}_2])$ and thus the $\mathbb{C}$-vector space $S_k \otimes |\det|^{k/2}_{\text{adélic}}$ is equipped with the non-degenerate $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant hermitian pairing

$$(\cdot,\cdot)\colon S_k \times S_k \to \mathbb{C},\ (f,g) \mapsto \int\limits_{[\mathrm{GL}_2]} \overline{\widetilde{\varphi_f}} \widetilde{\varphi}_g.$$

  - This implies by passing to orthogonal complements that each irreducible $\mathrm{GL}_2(\mathbb{A}_f)$-subquotient of S_k is already a direct summand.
  - Using Zorn's lemma each smooth non-zero $\mathrm{GL}_2(\mathbb{A}_f)$-representation V has an irreducible subquotient. Indeed, without loosing generality one may assume that V is generated by some $v \in V$. Then one applies Zorn's lemma to the set of submodules not containing v.

We need the following result by Shalika and Piatetski-Shapiro.

Theorem 5.5 (Shalika/Piatetski-Shapiro)**.** *For each finite set S of primes and each system of Hecke eigenvalues $\{\tilde{a}_p, \tilde{b}_p\}_{p \notin S}$ there exists at most one irreducible subrepresentation $V \subseteq S_k$ with this (equivalence class of) system of Hecke eigenvalues.*

- This is a special case of the "strong multiplicty one theorem" for cuspidal automorphic representations for GL_n, cf. [GH19, Theorem 11.7.2].[10]
- We can record the following consequence: Assume that $0 \neq f \in S_k$ is fixed by $K(m)$, and an eigenvector for the $\tilde{T}_p, \tilde{S}_p-$operators for $p \nmid m$. Then the $\mathrm{GL}_2(\mathbb{A}_f)$-representation

$$V := \langle f \rangle_{\mathrm{GL}_2(\mathbb{A}_f)}$$

  generated by f is irreducible.
  - Namely, each non-zero $v \in V$ yields a system of Hecke eigenvalues, which is equivalent to the one for f. By strong multiplicity one, they have to lie in the same subspace.

Now we will relate the Hecke operators $\tilde{T}_p, \tilde{S}_p$ to their classical counterparts acting on modular forms, and describe the decomposition of S_k via the newforms of Atkin/Lehner.

The decomposition of S_k indexed by newforms:

- By a result of Atkin-Lehner/Casselman (cf. [Cas73]) for each irreducible $\mathrm{GL}_2(\mathbb{A}_f)$-subrepresentation $V \subseteq S_k$, there exists some $m \in \mathbb{N}$ and some $0 \neq v \in V$ such that v is fixed by the subgroup

$$K_1(m) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0, d \equiv 1 \bmod m\} \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$$

- Set

$$\Gamma_1(m) := \mathrm{GL}_2(\mathbb{Z}) \cap K_1(m).$$

- Remarks:

[10]More precisely, we deduce it from the mentioned stronger result for $L^2_{\text{cusp}}([\mathrm{GL}_2])$ via the density of cuspidal automorphic forms, [GH19, Theorem 6.5.1.], and our embedding $S_k \otimes |\det|^{k/2}_{\text{adélic}}$.

  - Our notation is different than in many sources, where $\Gamma_1(m) \subseteq \mathrm{SL}_2(\mathbb{Z})$. This clash of notation is harmless as
$$\Gamma_1(m)\backslash\mathbb{H}^{\pm} \cong (\Gamma_1(m) \cap \mathrm{SL}_2(\mathbb{Z}))\backslash\mathbb{H}.$$
  - The minimal m for which there exists such a non-zero $v \in V$ is called the conductor of V. In this case, v is unique up to a scalar (and called the newform of the representation).
  - More details can be found in [Del73].
- We noted already that the morphism
$$\begin{array}{ccc} \Gamma_1(m)\backslash\mathbb{H}^{\pm} & \xrightarrow{\simeq} & \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K_1(m) \times \mathbb{H}^{\pm}) \\ [z] & \mapsto & [(1,z)] \end{array}$$
is an isomorphism in Section 3. Of course, the isomorphism is compatible with $\omega^{\otimes k}$.
- In particular,
$$M_k(\Gamma_1(m)) \cong M_k(K_1(m))$$
(and both agree with the space of classical modular forms for $\Gamma_1(m) \cap \mathrm{SL}_2(\mathbb{Z})$).
- Let us make this isomorphism more explicit and pick $\tilde{f} \in M_k(K_1(m))$.
- Let $f \in M_k(\Gamma_1(m))$ be the section corresponding to $\tilde{f}$.
- Then
$$\tilde{f}(g,z) = \tilde{f}(\gamma, z) = j(\gamma^{-1}, z)^{-k} f(\gamma^{-1}z),$$
where $g = \gamma h$ with $\gamma \in \mathrm{GL}_2(\mathbb{Q}), h \in K_1(m)$.
- Note that we are using that $\mathrm{GL}_2(\mathbb{Q})K_1(m) = \mathrm{GL}_2(\mathbb{A}_f)$.
- Fix some prime p with $p \nmid m$.
- We have the two operators $\tilde{T}_p, \tilde{S}_p$ acting on $M_k(K_1(m))$. Thus, by transport of structure they act on $M_k(\Gamma_1(m))$.
- Let us make this action of $\tilde{S}_p$ on $M_k(\Gamma_1(m))$ explicit.
- For a matrix $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ we denote by
$$\gamma_p \in \mathrm{GL}_2(\mathbb{A}_f)$$
the element whose component at p is $\gamma \in \mathrm{GL}_2(\mathbb{Q}) \subseteq \mathrm{GL}_2(\mathbb{Q}_p)$, and the identity matrix otherwise. This element should not be confused with
$$\gamma_{\mathrm{diag}} \in \mathrm{GL}_2(\mathbb{A}_f),$$
by which we mean the element obtained by diagonally embedding $\mathrm{GL}_2(\mathbb{Q}) \subseteq \mathrm{GL}_2(\mathbb{A}_f)$.
- The action of $\tilde{S}_p$ on $M_k(K_1(m))$ is easy to write down, but to describe it on $M_k(\Gamma_1(m)$ we have to find an expression
$$\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}_p = \gamma_{\mathrm{diag}} h, \text{ with } \gamma \in \mathrm{GL}_2(\mathbb{Q}), \quad h \in K_1(m).$$
- Because p, m are prime there exists $a, b \in \mathbb{Z}$, such that
$$A := \begin{pmatrix} a & b \\ m & p \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$
- Then
$$A_{\mathrm{diag}} \cdot \begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix}_{\mathrm{diag}} \cdot \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}_p \in K_1(m),$$

where we used the notation $(-)_{\mathrm{diag}}, (-)_p$ introduced before.

- In other words, we can take
$$\gamma := \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} A^{-1}$$
- With this choice of γ the operator $\tilde{S}_p$ acts on $M_k(\Gamma_1(m))$ via
$$f \mapsto (z \mapsto j(\gamma^{-1}, z)^{-k} f(\gamma^{-1} z)).$$
- The group
$$\Gamma_0(m) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 1 \bmod m \}$$
normalizes $\Gamma_1(m)$ and thus there is an action of $\Gamma_0(m)/\Gamma_1(m) \cong (\mathbb{Z}/m)^\times$ via
$$(B, f) \mapsto j(B, z)^{-k} f(Bz).$$
for $B \in \Gamma_0(m), f \in M_k(\Gamma_1(m))$.
- Thus, we obtain a decomposition
$$M_k(\Gamma_1(m)) = \bigoplus_{\chi\colon (\mathbb{Z}/m)^\times \to \mathbb{C}^\times} M_k(\Gamma_0(m), \chi)$$
with $M_k(\Gamma_0(m), \chi)$, by definition, the space of modular forms for $\Gamma_0(m)$ with nebentypus χ, i.e., modular forms for $\Gamma_1(m)$ satisfying
$$f(\frac{az+b}{cz+d}) = \chi(d)(cz+d)^k f(z) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m).$$
- For each Dirichlet character $\chi\colon (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$ there exists a unique adélic character
$$\psi_\chi\colon \mathbb{Q}^\times \backslash \mathbb{A}^\times / (1 + m\widehat{\mathbb{Z}}) \cap \widehat{\mathbb{Z}}^\times \to \mathbb{C}^\times,$$
such that $\psi_\chi(r) = r^{-k}$ for $r \in \mathbb{R}_{>0}$.
- We obtain that
$$M_k(\Gamma_0(m), \chi) \cong M_k(K_1(m), \psi_\chi),$$
where the RHS denotes the ψ_χ-eigenspace for the action of $\mathbb{A}_f^\times$ (the center of $\mathrm{GL}_2(\mathbb{A}_f)$) on $M_k(K_1(m))$.
- With this terminology we can finish the description of the operator $\tilde{S}_p$ on $M_k(\Gamma_1(M))$.
- Namely, the action of $\tilde{S}_p$ is given on the subspace $M_k(\Gamma_0(m), \chi) \subseteq M_k(\Gamma_1(m))$ by the action of
$$\gamma^{-1} = \begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix} A$$
and $\begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix}$ acts by p^k and $A \in \Gamma_0(m)$ by $\chi(p)$.
- Let us describe now the action (by transport of structure) of $\tilde{T}_p$ on $M_k(\Gamma_0(m), \chi)$.
- We have
$$\mathrm{GL}_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p) = \coprod_{j=0}^{p-1} \begin{pmatrix} p & j \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p) \coprod \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p).$$

- We again have to find factorizations of (the shown representatives of) the double cosets into
$$\gamma \cdot h, \quad \gamma \in \mathrm{GL}_2(\mathbb{Q}), \quad h \in K_1(m).$$
- We can take the factorization
$$\begin{pmatrix} p & j \\ 0 & 1 \end{pmatrix}_p = \begin{pmatrix} p & j \\ 0 & 1 \end{pmatrix}_{\mathrm{diag}} \cdot h$$
for $j = 0, \ldots, p-1$, with $h \in K_1(m)$, i.e.,
$$\gamma^{-1} = \begin{pmatrix} p^{-1} & -p^{-1}j \\ 0 & 1 \end{pmatrix}.$$
- For
$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$$
we find, similar to the argument above,
$$\gamma^{-1} = \begin{pmatrix} a & b \\ m & p \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \cdot \begin{pmatrix} a & p^{-1}b \\ pm & p \end{pmatrix}$$
as desired.
- Putting these together we obtain
$$\tilde{T}_p(f) = p^k \chi(p) f(pz) + \sum_{j=0}^{p-1} f(\frac{z-j}{p}).$$
- If the modular form with nebentypus $f \in M_k(\Gamma_0(m), \chi)$ is written in Fourier expansion
$$f(z) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i z},$$
then more explicitly
$$\tilde{T}_p(f)(z) = \sum_{n=0}^{\infty} p a_{np} q^n + \sum_{n=0}^{\infty} p^k \chi(p) a_n q^{pn}.$$
- This follows from the fact
$$\sum_{j=0}^{p-1} e^{2\pi i \frac{-jn}{p}} = \begin{cases} p, & p|n \\ 0, & \text{otherwise.} \end{cases}$$
- In particular, if $f \in M_k(\Gamma_0(m), \chi)$ is an eigenvector for $\tilde{T}_p$ and $a_1 = 1$, then (by considering the coefficient in front of q^1)
$$\widetilde{T_p}(f) = p a_p f.$$
This motivates to define
$$T_p = \frac{1}{p} \tilde{T}_p,$$
which is the classical Hecke operator acting on modular forms (like in [DS05, Section 5.2.]). In other words, the Hecke eigenvalues for the T_p-operators are the Fourier coefficients of the normalized eigenform.

We can now prove the desired theorem on decomposing the $\mathrm{GL}_2(\mathbb{A}_f)$-representation S_k.

Theorem 5.6. *The* $\mathrm{GL}_2(\mathbb{A}_f)$*-representation* S_k *decomposes into irreducibles as*

$$S_k \cong \bigoplus_f \langle f\rangle_{\mathrm{GL}_2(\mathbb{A}_f)}$$

with f *running through the set of (normalized) newforms for* $\Gamma_0(N)$ *with nebentypus* χ *for varying* N *and Dirichlet characters* $\chi\colon (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$*.*

We don't give the precise definition of a newform, but refer to [DS05, Section 5.8.]. Roughly, a newform is an eigenform realising the minimal N among all eigenforms with an equivalent system of Hecke eigenvalues.

Proof. After the above preparation we can now quote [DS05, Proposition 5.8.4], [DS05, Theorem 5.8.2], which imply that for each system of Hecke eigenvalues appearing in S_k there exists a unique normalized (i.e., in Fourier expansion $a_1 = 1$) newform with equivalent system of Hecke eigenvalues. □

In other textbooks, e.g., [Gel75, Proposition 3.1], automorphic representations are associated directly to some $f \in M_k(\Gamma_0(m), \chi)$, i.e., without introducing the space M_k, and the $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant embeddings

$$M_k \to C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})),\ f(g,z) \mapsto \varphi_f(g,g_\infty) = j(g_\infty, i)^{-k} f(g, g_\infty i)$$

resp.

$$S_k \otimes |\det|^{k/2}_{\text{adélic}} \to L^2([\mathrm{GL}_2]),\ f \mapsto \widetilde{\varphi}_f(g,g_\infty) = |\det(g,g_\infty)|^{k/2}_{\text{adélic}}\varphi_f.$$

We note that under the isomorphism

$$M_k(\Gamma_1(m)) \cong M_k(K_1(m))$$

mentioned above, our formulas specialize to the one in [Gel75, Proposition 3.1.]. Moreover, the maybe unexpected factor $p^{k/2-1}$ appearing in [Gel75, Lemma 3.7.] has an easy explanation by the appearance of the twist by $|\det|^{k/2}_{\text{addélic}}$ and the passage from (our) $\widetilde{T}_p$ to the classical Hecke operator $T_p = \frac{1}{p}\widetilde{T}_p$.

6. Langlands reciprocity for newforms

Last time:

- We proved the decomposition
$$S_k \cong \bigoplus_f \langle f\rangle_{\mathrm{GL}_2(\mathbb{A}_f)}$$
into irreducible $\mathrm{GL}_2(\mathbb{A}_f)$-representations.
- Here f is running through the set of (normalized) newforms of weight k for $\Gamma_0(N)$ with nebentypus χ for varying N and varying Dirichlet characters
$$\chi\colon (\mathbb{Z}/N)^\times \to \mathbb{C}^\times.$$
- In course, of proving the decomposition we associated to each irreducible, admissible $\mathrm{GL}_2(\mathbb{A}_f)$-representation π a system of Hecke eigenvalues
$$\{\tilde{a}_p(\pi), \tilde{b}_p(\pi)\}_{p\notin S}$$
for $S := \{p \text{ prime with } \pi^{\mathrm{GL}_n(\mathbb{Z}_p)} = 0\}$. In fact, $\tilde{a}_p(\pi)$ resp. $\tilde{b}_p(\pi)$ was defined to be the eigenvalue of the double coset
$$\mathrm{GL}_2(\mathbb{Z}_p)\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p)$$
resp.
$$\mathrm{GL}_2(\mathbb{Z}_p)\begin{pmatrix} p & 0 \\ 0 & p\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p)$$
when acting on a non-zero vector fixed by $\mathrm{GL}_2(\mathbb{Z}_p)$.
- If $\pi = \langle f\rangle_{\mathrm{GL}_2(\mathbb{A}_f)}$ for a normalized newform $f \in S_k(\Gamma_0(N), \chi)$, then $S = \{\text{prime divisors of } N\}$ and
$$\tilde{a}_p(\pi) = pa_p, \quad \tilde{b}_p(\pi) = p^k\chi(p)$$
if $p \notin S$, where $f(q) = \sum\limits_{n=1}^{\infty} a_n q^n$ is the Fourier expansion of f.

In this lecture, we will introduce traces of Frobenii for ℓ-adic Galois representations (which are the "Galois-theoretic" counter part of a system of Hecke eigenvalues). Then we want to state what the Langlands program predicts for S_k.

Traces of Frobenii for ℓ-adic representations:

- Let ℓ be some prime.
- $\overline{\mathbb{Q}}_\ell = \varinjlim\limits_{E/\mathbb{Q}_\ell} E$ is a topological field via colimit topology.
- Let W be a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space.
- Let $\sigma\colon G_\mathbb{Q} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(W)$ be a continuous representation, also called an "ℓ-adic Galois representation".
- We recall that for each prime p there exists an embedding
$$G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to G_\mathbb{Q},$$
well-defined up to conjugacy in $G_\mathbb{Q}$.
- The local absolute Galois group $G_{\mathbb{Q}_p}$ sits in the exact sequence
$$1 \to I_p \to G_{\mathbb{Q}_p} \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = (\mathrm{Frob}_p^{\mathrm{geom}})^{\widehat{\mathbb{Z}}} \to 1$$
with $I_p \subseteq G_{\mathbb{Q}_p}$ the inertia subgroup, and
$$\mathrm{Frob}_p^{\mathrm{arith}}\colon \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p,\ x \mapsto x^p$$

the arithmetic Frobenius at p.

- The inverse $\mathrm{Frob}_p^{\mathrm{geom}} := \mathrm{Frob}_p^{\mathrm{arith}^{-1}}$ is called the geometric Frobenius.
- Let S be a finite set of primes. We say σ is unramified outside S if $\sigma(I_p) = 1$ for all $p \notin S$.
- If σ unramified outside S and $n := \dim_{\overline{\mathbb{Q}}_\ell} W$, we can associate to σ a system
$$\{c_{1,p}, \ldots, c_{n,p}\}_{p \notin S}$$
of elements $c_{1,p}, \ldots, c_{n,p} \in \overline{\mathbb{Q}}_\ell$. Namely,
$$c_{i,p} := \mathrm{Tr}(\sigma(\mathrm{Frob}_p^{\mathrm{arith}})|\Lambda^i W).$$
- Note: The $c_{i,p}$ are well-defined because $\sigma(I_p) = 1$.
- Note: The $c_{i,p} = \mathrm{Tr}(\sigma(\mathrm{Frob}_p^{\mathrm{geom}})|\Lambda^i W)$ depend only on p (and not on the embedding $G_{\mathbb{Q}_p} \subseteq G_{\mathbb{Q}}$ as the trace is invariant under conjugation.
- If W is semisimple, then σ is determined by the
$$\{c_{1,p}, \ldots, c_{n,p}\}_{p \notin S}$$
for each finite set of primes S such that W is unramified outside S, cf. [Ser97, I-10].
- In fact, the collection $\{c_{1,p}\}_{p \notin S}$ is sufficient (plus the assumption that W semisimple and unramified outside S).
- Indeed:
  - Consider the group algebra $\Lambda := \overline{\mathbb{Q}}_\ell[G_{\mathbb{Q},S}]$, where $G_{\mathbb{Q},S}$ is the quotient of $G_{\mathbb{Q}}$ by the closure of the subgroup generated by the $I_p, p \notin S$.
  - Assume that W, W' are two semisimple, continuous $G_{\mathbb{Q},S}$-representations with
  $$\mathrm{Tr}(\mathrm{Frob}_p^{\mathrm{arith}}|W) = \mathrm{Tr}(\mathrm{Frob}_p^{\mathrm{arith}}|W')$$
  for all $p \notin S$.
  - By Chebotarev density and continuity this implies
  $$\mathrm{Tr}(\lambda|W) = \mathrm{Tr}(\lambda|W'),$$
  for all $\lambda \in \Lambda$.
  - For irreducible, pairwise non-isomorphic Λ-modules $W_1, \ldots, W_m$ and each $i = 1, \ldots, m$, there exists $\mu_i \in \Lambda$, such that
  $$\mu_i = 1$$
  on W_i, but
  $$\mu_i = 0$$
  on $W_j, j \neq i$ (this is a version of the Chinese remainder theorem).
  - Write
  $$W = \bigoplus_i^m W_i^{\oplus n_i}, \quad W' = \bigoplus_{i=1}^m W_i^{\oplus n_i'}$$
  in isotypic components with $W_1, \ldots, W_m$ irreducible, pairwise non-isomorphic, and $n_i, n_i' \in \mathbb{N}$, possibly zero.
  - Then
  $$n_i \dim_{\overline{\mathbb{Q}}_\ell} W_i = \mathrm{Tr}(\mu_i|W) = \mathrm{Tr}(\mu_i|W') = n_i' \dim_{\overline{\mathbb{Q}}_\ell} W_i$$
  for all $i = 1, \ldots, m$, i.e., $n_i = n_i', i = 1, \ldots, m$ and $W \cong W'$ as desired.
- This statement can be seen as an analog of "strong multiplicity one" for cuspidal automorphic representations for GL_n mentioned last time.

- It was important that we considered representations with coefficients in characteristic 0, i.e., it is wrong that the traces $c_{i,p}$ determine the (semisimple) representation. By the Brauer–Nesbitt theorem it is still true that the full system $\{c_{1,p}, \ldots, c_{n,p}\}_{p \notin S}$ determines a finite dimensional semisimple representation uniquely.

The traces of Frobenius elements contain informations of high arithmetic significance. The following discussion is very important for understanding the arithmetic implications of the Langlands program.

Arithmetic significance of the traces of Frobenii:

- Let $\sigma\colon G_{\mathbb{Q}} \to \mathrm{GL}(W)$ be an ℓ-adic representation, unramified outside S.
- The traces
$$c_{1,p} := \mathrm{Tr}(\sigma(\mathrm{Frob}_p^{\mathrm{arith}})|W)$$
encode significant arithmetic information.
- For example, assume that $F/\mathbb{Q}$ is finite Galois extension with Galois group $\mathrm{Gal}(F/\mathbb{Q}) \cong S_3$, and that
$$\sigma\colon G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(F/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$
is inflated from the unique irreducible, 2-dimensional representation of S_3. Set
$$S := \{p \text{ ramified in } F\}$$
and for $p \notin S$
$$\mathrm{Frob}_{F,p}^{\mathrm{arith}}$$
as the image of $\mathrm{Frob}_p^{\mathrm{arith}} \in G_{\mathbb{Q},S}$ in $\mathrm{Gal}(F/\mathbb{Q}) \cong S_3$.
- Then:
  *
$$\begin{array}{cc} & \mathrm{Tr}(\mathrm{Frob}_p^{\mathrm{arith}}|W) = 2 \\ \Leftrightarrow & \mathrm{Frob}_{F,p}^{\mathrm{arith}} \in \{1\} \\ \Leftrightarrow & p \text{ splits completely in } \mathcal{O}_F \end{array},$$
  *
$$\begin{array}{cc} & \mathrm{Tr}(\mathrm{Frob}_p^{\mathrm{arith}}|W) = 0 \\ \Leftrightarrow & \mathrm{Frob}_{F,p}^{\mathrm{arith}} \in \{(1,2),(1,3),(2,3)\} \\ \Leftrightarrow & p \text{ splits into three distinct primes in } \mathcal{O}_F \end{array},$$
  *
$$\begin{array}{cc} & \mathrm{Tr}(\mathrm{Frob}_p^{\mathrm{arith}}|W) = -1 \\ \Leftrightarrow & \mathrm{Frob}_{F,p}^{\mathrm{arith}} \in \{(1,2,3),(1,3,2)\} \\ \Leftrightarrow & p \text{ splits into two distinct primes in } \mathcal{O}_F \end{array}.$$
- Thus: Knowledge of traces of Frobenii for all ℓ-adic Galois representations (with finite image) implies knowledge of the decomposition of unramified primes in *all* finite extensions of $\mathbb{Q}$.
- Other examples of ℓ-adic representations are the étale cohomology groups
$$H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$$
for proper, smooth schemes X over $\mathbb{Q}$, and $i \geq 0$.
- The $G_{\mathbb{Q}}$-action is induced by functoriality of $H^i_{\text{ét}}(-, \overline{\mathbb{Q}}_\ell)$ from the (right) action of $G_{\mathbb{Q}}$ on
$$X_{\overline{\mathbb{Q}}} = X \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\overline{\mathbb{Q}}).$$

- Proper, smooth base change together with the Grothendieck-Lefschetz trace formula imply that if X has good reduction at p and $p \neq \ell$, then $H^i_{\text{ét}}(X_{\overline{Q}}, \overline{\mathbb{Q}}_\ell)$ is unramified at p for $i \geq 0$ and
$$\sum_{i=0}^{2\dim(X)} (-1)^i \mathrm{Tr}(\mathrm{Frob}_p^{\mathrm{geom}} | H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)) = \sharp \mathcal{X}(\mathbb{F}_p),$$
where $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z}_{(p)})$ is a proper, smooth model of X at p.

From the arithmetic point of view we can now state one major point of the Langlands program.

The whole point of Langlands reciprocity:

- Traces of Frobenii are supposed to match Hecke eigenvalues!
- Thus, arithmetic information is expected to be encoded in automorphic information, which might be more accessible.
- The following example was taken from a post of Matthew Emerton. More insightful posts by him can be found via his webpage: `http://www.math.uchicago.edu/~emerton/` .
  - Let F be the splitting field of $x^3 - x - 1$.
  - Then F is the Hilbert class field of $\mathbb{Q}(\sqrt{-23})$, and Galois over $\mathbb{Q}$ with Galois group S_3.
  - Consider as before the associated 2-dimensional, irreducible Galois representation $\sigma\colon G_{\mathbb{Q}} \to \mathrm{Gal}(F/\mathbb{Q}) \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$.
  - Hecke proved that there exists a normalized newform $f(q) = \sum\limits_{i=1}^{\infty} a_n q^n$, such that
  $$a_p = \mathrm{Tr}(\sigma(\mathrm{Frob}_p^{\mathrm{arith}}))$$
  for all primes p with $p \nmid 23$.
  - One can make f more explicit: Considering ramification, f must lie in $S_1(\Gamma_0(23), \chi)$ with $\chi\colon (\mathbb{Z}/23)^\times \to \mathbb{C}^\times$ the quadratic character determining the quadratic subfield
  $$\mathbb{Q}(\sqrt{-23}) \subseteq \mathbb{Q}(\zeta_{23}).$$
  People knowing modular forms (or a database like LMFDB) will tell us that we must have
  $$f(q) = q \prod_{i=1}^{\infty} (1 - q^i)(1 - q^{23i}).$$
  - We can now calculate the developement of f (or look it up in LMFDB):
  $$f(q) = q - q^2 - q^3 + \ldots + q^{58} + 2q^{59} + \ldots$$
  and deduce how primes decompose in F, e.g., 59 splits completely!
- Perhaps the most famous example how modularity of traces of Frobenii was used was the proof of Fermat's last theorem.
- Namely, Fermat's theorem was proved by Wiles/Taylor using the following strategy (initiated by Frey, and complemented by Serre, Ribet):
  - Assume $u^p + v^p + w^p = 0$ with $u, v, w \in \mathbb{Q}$, $uvw \neq 0$ and $p \geq 3$.
  - Consider the elliptic curve E over $\mathbb{Q}$ with (affine) Weierstraß equation
  $$y^2 = x(x + u^p)(x - v^p)$$

(after possibly manipulating u, v, w a bit).
- Consider the dual σ of the Galois representation
$$G_{\mathbb{Q}} \to \mathrm{GL}(H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)) \cong \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell).$$
- Show that σ is modular, i.e., there exists a normalized newform $f_1(q) = \sum_{n=1}^{\infty} a_n q^n$ of weight 2 such that
$$\mathrm{Tr}(\sigma(\mathrm{Frob}^{\mathrm{arith}})) = a_p$$
for almost all primes p.
- Using a theorem of Ribet, one concludes that f_1 must be congruent to a normalized newform f_2 in
$$S_2(\Gamma_0(2)),$$
i.e., the Fourier coefficients of f_1, f_2 are algebraic integers and congruent modulo some prime.
- But $S_2(\Gamma_0(2)) = 0$, and thus f_2 cannot exist.
- This yields the desired contradiction.
- For more details, cf. [Wil95], [Rib90].

Now, we state what is expected on the relation between Galois representations and cusp forms.

Langlands reciprocity for newforms:

- Fix a prime ℓ and an isomorphism
$$\iota\colon \overline{\mathbb{Q}}_\ell \cong \mathbb{C}.$$
- For newforms the Langlands program combined with the Fontaine-Mazur conjecture predicts a bijection
$$\mathrm{LL}\colon \mathcal{A}_{\mathrm{mod}} \xrightarrow{\simeq} \mathcal{G}_{\mathrm{mod}}$$
from the set
$$\mathcal{A}_{\mathrm{mod}} := \{\text{irreducible } \mathrm{GL}_2(\mathbb{A}_f)\text{-subrepresentations } \pi \subseteq \bigoplus_{k\geq 1} S_k\},$$
to a certain set
$$\mathcal{G}_{\mathrm{mod}}$$
consisting of irreducible, 2-dimensional Galois representations
$$\sigma\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$
- LL should satisfy that for all $\pi \in \mathcal{A}_{\mathrm{mod}}$ with $\sigma := \mathrm{LL}(\pi)$ we have
$$\iota(\mathrm{Tr}(\sigma(\mathrm{Frob}_p^{\mathrm{arith}}))) = a_p(\pi), \quad \iota(\det(\sigma(\mathrm{Frob}_p^{\mathrm{arith}}))) = b_p(\pi)$$
for p outside some specified finite set S of primes.

In a previous version I considered the geometric Frobenius, which was wrong.

- Here, the eigenvalues $a_p(\pi)$ resp. $b_p(\pi)$ are used and not the $\tilde{a}_p(\pi), \tilde{b}_p(\pi)$.
- Note: Such a bijection LL is uniquely determined (by the respective multiplicity one theorems), if it exists.
- Conjecturally, the set is S specified explicitly. Recall that a prime p unramified for σ resp. π if
$$\sigma(I_p) = 1$$
resp.
$$\pi^{\mathrm{GL}_2(\mathbb{Z}_p)} \neq 0.$$

- Then a prime $p \neq \ell$ is conjecturally unramified for π if and only if it is unramified for $\sigma := \mathrm{LL}(\pi)$.
- The matching of Hecke eigenvalues states then more precisely that $\sigma(\mathrm{Frob}_p^{\mathrm{arith}})$ should have characteristic polynomial

$$X^2 - \iota^{-1}(a_p(\pi))X + \iota^{-1}(b_p(\pi))$$

  for each unramified prime $p \neq \ell$ for π.
- It is the next aim of the course to indicate how the map LL can be constructed, i.e., how to associate Galois representations to normalized newforms. For this we follow results of Deligne, cf. [Del71b], and Deligne/Serre, cf. [DS74].

Let us make more precise which set of Galois representations is expected to be associated to newforms.

The set $\mathcal{G}_{\mathrm{mod}}$:

- The set $\mathcal{G}_{\mathrm{mod}}$ was defined by Fontaine/Mazur in relation with their remarkable conjecture on geometric Galois representations, cf. [FM95].
- Let $\sigma\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be an irreducible, 2-dimensional ℓ-adic representation.
- Then by definition $\sigma \in \mathcal{G}_{\mathrm{mod}}$ if and only if
  - σ unramified at almost all p, i.e., $\sigma(I_p) = 1$ for p outside some finite set of primes S.
  - σ is odd, i.e., for each complex conjugation $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \subseteq G_{\mathbb{Q}}$ we have

$$\det(\sigma(c)) = -1.$$

  - The restriction

$$\sigma_\ell := \sigma_{|G_{\mathbb{Q}_\ell}} \colon G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

    is *de Rham* with Hodge-Tate weights $0, \omega$ for some $\omega \in \mathbb{N}$ (note that $\omega = 0$ is allowed).
- If $\sigma = \mathrm{LL}(\pi)$ with $\pi \subseteq S_k$, then conjecturally $\omega = k - 1$.
- Using the work of many people (Wiles, Kisin, Emerton, Skinner-Wiles, Pan,...) any $\sigma \in \mathcal{G}_{\mathrm{mod}}$ with distinct Hodge-Tate weights is modular if $\ell \geq 5$, cf. [Pan19, Theorem 1.0.4.].
- If $\sigma \in \mathcal{G}_{\mathrm{mod}}$ has finite image, then σ is modular, cf. [PS16], [GH19, Section 13.4.] (and the references therein).
- Conjecturally: HT-weights 0 $\Leftrightarrow$ image of σ finite.

We have to make a short digression on the important notion of being de Rham.

de Rham representations:

- Let p be a prime (the previous ℓ).
- Let $K/\mathbb{Q}_p$ a discretely valued, non-archimedean extension with perfect residue field (e.g., $K/\mathbb{Q}_p$ finite).
- Let $\sigma\colon G_K := \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$ be a continuous representation.
- Then σ has coefficients in some finite extension $E/\mathbb{Q}_p$ (by compactness of G_K as $\overline{\mathbb{Q}}_p$ carries the colimit topology here).
- We obtain a $\mathbb{Q}_p$-linear representation, a "local p-adic Galois representation",

$$\rho\colon G_K \to \mathrm{GL}_m(E) \subseteq \mathrm{GL}_n(\mathbb{Q}_p),$$

  where $n := m \cdot \dim_{\mathbb{Q}_p} E$.

- Thus, we can change our setup and let V be a finite dimensional $\mathbb{Q}_p$-vector space, and $\rho\colon G_K \to \mathrm{GL}(V)$ a continuous representation.
- Then V is called de Rham if
$$\dim_K(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} = \dim_{\mathbb{Q}_p} V$$
for a certain field extension B_{dR} of K with G_K-action, cf. [BC09, Section 6].
- Namely: B_{dR} is Fontaine's field of p-adic periods, cf. [Fon94], [BC09, Definition 4.4.7]. Abstractly,
$$B_{\mathrm{dR}} \cong \mathbb{C}_K((t)),$$
where $\mathbb{C}_K$ is the completion of $\overline{K}$ for its p-adic valuation (this topology is coarser than the colimit topology on $\overline{K}$).
- B_{dR} is a discretely valued field, with residue field $\mathbb{C}_K$, the deduced decreasing filtration $\mathrm{Fil}^\bullet B_{\mathrm{dR}}$ is G_K-stable and as $\mathbb{C}_K$-semilinear G_K-representations
$$\mathrm{gr}^\bullet B_{\mathrm{dR}} \cong \bigoplus_{j\in\mathbb{Z}} \mathbb{C}_K(j),$$
where $\mathbb{C}_K(j) := \mathbb{C}_K \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(j)$, with
$$\mathbb{Z}_p(1) := T_p\mu_{p^\infty}(\mathbb{C}_K)$$
the p-adic Galois representation associated to the cyclotomic character
$$\chi_{\mathrm{cyc}}\colon G_K \to \mathbb{Z}_p^\times \cong \mathrm{Aut}(\mu_{p^\infty}(\mathbb{C}_K)).$$
- An important theorem of Tate, cf. [BC09, Theorem 2.2.7], states the following: Let $\chi\colon G_K \to \mathbb{Q}_p^\times$ be a character. Then
$$H^0(G_K, \mathbb{C}_K(\chi)) = \begin{cases} K, & \text{if } \chi_{|I_K} \text{ has finite image} \\ 0, & \text{otherwise }, \end{cases}$$
here $I_K \subseteq G_K$ is the inertia subgroup.
- In particular:
$$H^0(G_K, \mathbb{C}_K(j)) = 0, \quad \text{if } j \neq 0,$$
as $\chi^j_{\mathrm{cyc}|I_K}$ has infinite image if $j \neq 0$.
- Conclusion: If $\dim_{\mathbb{C}} V = 1$, then V is de Rham if and only if V is isomorphic to $\chi \cdot \chi^j_{\mathrm{cyc}}$ for some character $\chi\colon G_K \to \mathbb{Q}_p^\times$ with I_K of finite order, and $j \in \mathbb{Z}$. In particular, there exists many representations, which are not de Rham, e.g., the characters $(\chi_{\mathrm{cyc}}^{p-1})^a$ with $a \in \mathbb{Z}_p \setminus \mathbb{Z}$.
- Let us mention some general facts on de Rham representations.
  - $\rho\colon G_K \to \mathrm{GL}(V)$ is de Rham if $\rho_{|G_{K'}}$ is de Rham for some finite extension K' of K, cf. [BC09, Proposition 6.3.8].
  - de Rham representations are stable under subquotients, duals and tensor products, cf. [BC09, Section 6.1].
- Let V be a de Rham representation of dimension n. Then
$$V \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong \mathbb{C}_K(j_1) \oplus \ldots \oplus \mathbb{C}_K(j_n)$$
as $\mathbb{C}_K$-semilinear G_K-representations, where $j_1, \ldots, j_n \in \mathbb{Z}$. The unordered collection $(j_1, \ldots, j_n)$ is called the collection of Hodge-Tate weights for V.

Thus, the condition of being de Rham puts some sort of "integrality" condition on a p-adic representation.

We close the lecture by indicating how the map LL will be constructed (if $k = 2$).

Strategy for constructing LL (roughly):

- Let $\pi \in \mathcal{A}_{\mathrm{mod}}$, with associated system of Hecke eigenvalues
$$\{a_p, b_p\}_{p \notin S}.$$
- Assume for simplicity that $\pi \subseteq S_2$, i.e., π is generated by some newform of weight 2.
- Then we will construct (something close to) a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant embedding
$$S_2 \hookrightarrow H^1(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}), \mathbb{C}) := \varinjlim_K H^1(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^{\pm}), \mathbb{C})$$
of S_2 into cohomology.
- The given isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ yields an isomorphism
$$H^1(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}), \mathbb{C}) \cong H^1(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}), \overline{\mathbb{Q}}_\ell)$$
of cohomology groups.
- As already mentioned for $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ a (sufficiently small) compact-open subgroup, the complex manifold
$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^{\pm})$$
is actually *algebraic*, i.e., given by $\tilde{X}_K(\mathbb{C})$ for some quasi-projective scheme $\tilde{X}_K \to \mathrm{Spec}(\mathbb{C})$.
- The étale comparison theorem implies
$$H^1(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^{\pm}), \overline{\mathbb{Q}}_\ell) \cong H^1_{\text{ét}}(\tilde{X}_K, \overline{\mathbb{Q}}_\ell).$$
- Now the miracle happens: For $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ a (sufficiently small) compact-open subgroup

the quasi-projective scheme $\tilde{X}_K \to \mathrm{Spec}(\mathbb{C})$ is canonically defined over $\mathrm{Spec}(\mathbb{Q})$!

  In other words, $\tilde{X}_K \cong X_K \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{C})$ for (compatible) quasi-projective schemes $X_K \to \mathrm{Spec}(\mathbb{Q})$.
- In particular,
$$H^1_{\text{ét}}(\tilde{X}_K, \overline{\mathbb{Q}}_\ell) \cong H^1_{\text{ét}}(X_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$$
by invariance of étale cohomology under change of algebraically closed base fields. But now the RHS carries an action of $G_{\mathbb{Q}}$!
- Look now at the $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$-module
$$H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) := \varinjlim_K H^1_{\text{ét}}(X_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell).$$
- Then we will check that the $G_{\mathbb{Q}}$-module
$$LL(\pi) := \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A}_f)}(\pi, H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell))$$
is 2-dimensional and that we can express the traces of Frobenii by Hecke eigenvalues. The dual of this 2-dimensional representation will then be the desired representation.
- Replacing $\overline{\mathbb{Q}}_\ell$ by some local system a similar strategy works for weight $k \geq 2$.

- However, this does not help for $k = 1$. Here one has to use that a weight 1 modular form can be congruent to some modular form of weight $k \geq 2$, and use the previous case.

7. Modular curves as moduli of elliptic curves (by Ben Heuer)

Last time:

- First rough sketch of construction of

$$\mathrm{LL}\colon \mathcal{A}_{\mathrm{mod}} \xrightarrow{\simeq} \mathcal{G}_{\mathrm{mod}}$$

$$\mathcal{A}_{\mathrm{mod}} := \{\text{irreducible } \mathrm{GL}_2(\mathbb{A}_f)\text{-subrepresentations } \pi \subseteq \bigoplus_{k\geq 1} S_k\},$$

$$\mathcal{G}_{\mathrm{mod}} := \{\sigma : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell),\ \text{unramified outside finite set } S, \text{ odd, de Rham above } \ell\}$$

- Crucial point for construction: For sufficiently small compact-open subgroups $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ the complex manifold

$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^{\pm})$$

  (1) is algebraic over $\mathrm{Spec}(\mathbb{C})$,
  (2) admits a canonical model X_K over $\mathrm{Spec}(\mathbb{Q})$.

  This gives rise to the Galois action on l-adic étale cohomology $H^1_{\text{ét}}(X_{K,\bar{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$.

Today:

- Explain why the algebraic model X_K exists.
- Reinterpret notions defined before in algebro-geometric terms: modular forms, compactification, q-expansions, adelic Hecke action, ...

Reminder on elliptic curves Reference: [Silverman: The Arithmetic...]

- Let K be any field. We have the following equivalent definitions:

  Definition 7.1. *An **elliptic curve** over K is equivalently a*
  - *connected smooth projective curve $E|K$ of genus 1 with a chosen point $O \in E(K)$.*
  - *connected smooth projective algebraic group $E|K$ of dimension 1.*
  - *non-singular plane cubic curve, defined by a Weierstraß equation*

  $$E : y^2 = x^3 + ax + b, \quad a, b \in K$$

  (if $\mathrm{char} K \notin \{2,3\}$*, otherwise more terms are required, cf.* [Del75]*).*

- Fact: The group scheme is automatically commutative (!).
- Fact: The non-singularity can be expressed as a condition on the discriminant:

$$E \text{ non-singular} \quad \Leftrightarrow \quad \Delta(a,b) = -16(4a^3 + 27b^2) \neq 0$$

  (if $\mathrm{char} K \notin \{2,3\}$).
- Fact: Given E, the Weierstraß equation is not unique. But the **j-invariant**

$$j(E) := j(a,b) = 1728 \frac{4a^3}{4a^3 + 27b^2}$$

  is independent of choice of Weierstraß equation.
- Fact: If K is algebraically closed, we have

$$E \cong E' \Leftrightarrow j(E) = j(E').$$

- More generally, we can replace $\mathrm{Spec}(K)$ by any base scheme S.

  Definition 7.2. *An elliptic curve over S is a proper smooth curve $E \to S$ with geometrically connected fibres of genus 1, together with a point $0 : S \to E$.*

- This can be thought of as family of elliptic curves parametrised by S.
- Again, this automatically has a group structure:

  Theorem (Abel)**.** *There is a unique way to endow an elliptic curve $E \to S$ with the structure of a commutative S-group scheme s.t. $0 =$ identity section.*

- Zariski-locally on S, one can describe E in terms of a Weierstraß equation.

  Definition 7.3. *A morphism of elliptic curves $E \to E'$ over S is a homomorphism of S-group schemes. An* ***isogeny*** *is an fppf-surjective homomorphism with finite flat kernel.*

  Example 7.4. For any $N \in \mathbb{Z}$, the group scheme structure gives the morphism "multiplication by N", which is denoted by $[N] : E \to E$.

- Fact: $[N]$ is finite flat of rank N^2. It is finite étale if N is invertible on S.
- In particular, $[N]$ is an isogeny.

  Definition 7.5. $E[N] := \ker[N] \subseteq E$*, the N-****torsion subgroup****. This is a finite flat closed subgroup scheme. It is finite étale if N is invertible on S.*

- More generally, for any finite flat subgroup scheme $D \subseteq E[N]$ of exponent N, $\exists$ unique elliptic curve E/D with isogeny $\varphi : E \to E/D$ with kernel D.
- $[N]$ induces a **dual isogeny** $\varphi^\vee : E/D \to E$ with kernel $E[N]/D$.

Elliptic curves over $K = \mathbb{C}$ Reference: [Diamond–Shurman: A first course...]

- Let $E|\mathbb{C}$ be an elliptic curve. Then $E(\mathbb{C})$ has the structure of a compact complex Lie group. As such, it can be canonically uniformised:
- Let Lie E be the tangent space at $0 \in E(\mathbb{C})$. Then Lie $E \cong \mathbb{C}$ and there is an isomorphism of complex Lie groups
$$E(\mathbb{C}) = \mathrm{Lie}\ E/H_1(E,\mathbb{Z}) \cong \mathbb{C}/\Lambda, \quad \text{where } \Lambda \cong \mathbb{Z}^2.$$
- For any $N \in \mathbb{Z}$, we have
$$E(\mathbb{C})[N] = (\tfrac{1}{N}\Lambda)/\Lambda \subseteq \mathbb{C}/\Lambda.$$
- Note: $E(\mathbb{C})[N] = (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2 = (\mathbb{Z}/N)^2$ is of rank N^2, as claimed.
- Upshot: $E(\mathbb{C})$ is a **1-dimensional complex torus**: A quotient of $\mathbb{C}$ by a lattice $\mathbb{Z}^2 \cong \Lambda \subseteq \mathbb{C}$ such that $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}$.
- morphisms of complex tori := morphisms of complex Lie groups

  Theorem. *There is an equivalence of categories*
$$\{\text{elliptic curves over } \mathbb{C}\} \to \{1\text{-dim complex tori}\}$$

- Essential surjectivity: Let $\Lambda \subseteq \mathbb{C}$ be a lattice as above. Define the Weierstraß $\wp$-function
$$\wp(z,\Lambda) = \frac{1}{z^2} + \sum_{\substack{w\in\Lambda \\ w\neq 0}} \left(\frac{1}{z^2} - \frac{1}{(z-w)^2}\right)$$
- Then $\wp(z) = \wp(z,\Lambda)$ is holomorphic on $\mathbb{C}\backslash\Lambda$ with complex derivative
$$\wp'(z) = -2\sum_{w\in\Lambda} \frac{1}{(z-w)^3}.$$

- This is clearly Λ-periodic $\rightsquigarrow \wp$ and $\wp'$ are meromorphic functions on $\mathbb{C}/\Lambda$.

  Theorem. *The functions $\wp$ and $\wp'$ are related by a Weierstraß equation*
  $$E_\Lambda : \wp'^2 = 4(\wp^3 + g_2(\Lambda)\wp + g_3(\Lambda))$$
  for explicit $g_2(\Lambda), g_3(\Lambda) \in \mathbb{C}$. We thus get an isomorphism of Lie groups
  $$(\wp, \wp') : \mathbb{C}/\Lambda \xrightarrow{\sim} E_\Lambda(\mathbb{C}).$$
- Sending $\mathbb{C}/\Lambda \mapsto E_\Lambda$ defines a quasi-inverse
  $$\{\text{1-dim complex tori}\} \to \{\text{elliptic curves over } \mathbb{C}\}.$$

Complex moduli spaces of elliptic curves

- We can now parametrise complex elliptic curves up to isomorphism.
- For this, we just have to determine when two lattices Λ_1, Λ_2 satisfy
  $$\mathbb{C}/\Lambda_1 \cong \mathbb{C}/\Lambda_2.$$
- Fact: Any homomorphism of complex tori is multiplication by $a \in \mathbb{C}$. Indeed, it lifts to a holomorphic homomorphism on the Lie algebras.

$\Rightarrow$ this happens iff $\Lambda_1 = a\Lambda_2$ for some $a \in \mathbb{C}^\times$.
- Can now uniformise as follows: Write $\Lambda = w_1\mathbb{Z} \oplus w_2\mathbb{Z}$.
- The condition $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}$ implies $w_1/w_2 \in \mathbb{C} \setminus \mathbb{R} = \mathbb{H}^\pm$.

$\Rightarrow$ For every complex torus E, there is $\tau \in \mathbb{H}^\pm$ such that
$$E \cong E_\tau := \mathbb{C}/\Lambda_\tau, \quad \Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$$
- Finally, for $\tau, \tau' \in \mathbb{H}^\pm$, we have
  $$\mathbb{Z} + \tau\mathbb{Z} = \mathbb{Z} + \tau'\mathbb{Z} \Leftrightarrow \exists \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}) : \gamma\tau = \frac{a\tau + b}{c\tau + d} = \tau'$$
- This shows: $\tau \mapsto E_\tau$ defines
  $$\mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^\pm \cong \{\text{complex tori}\}/\sim \cong \{\text{elliptic curves over } \mathbb{C}\}/\sim$$
  This gives our space from earlier,
  $$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} = \mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^\pm = \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm) \text{ for } K = \mathrm{GL}_2(\hat{\mathbb{Z}})$$
  an interpretation as a "moduli space" (so far, in a weak sense):
  $$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} = \{\text{elliptic curves over } \mathbb{C}\}/\sim$$
- $\mathbb{C}$ algebraically closed $\Rightarrow$ j-invariant defines bijection (in fact biholomorphism)
  $$j : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \xrightarrow{\sim} \mathbb{C}.$$
- Via the covering map $\mathbb{H}^\pm \to \mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^\pm$, we get from the above:
  $$\begin{aligned}\mathbb{H}^\pm =& \{\text{complex tori with ordered basis } \alpha : \mathbb{Z}^2 \xrightarrow{\sim} \Lambda\}/\sim \\ =& \{\text{elliptic curves } E \text{ over } \mathbb{C} \text{ with } \alpha : \mathbb{Z}^2 \xrightarrow{\sim} H_1(E, \mathbb{Z})\}/\sim\end{aligned}$$
- What about moduli interpretations of other levels? Recall for $N \in \mathbb{N}$, we had the level $K = K(N)$,
  $$X_K := \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm) = \coprod \Gamma(N)\backslash\mathbb{H}^\pm,$$
  where
  $$\Gamma(N) = \{\gamma \in \mathrm{GL}_2(\mathbb{Z}) | \gamma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod N\}.$$

- These are precisely the automorphisms of Λ that preserve the subgroup
$$E_\Lambda[N] = \tfrac{1}{N}\Lambda/\Lambda \subseteq \mathbb{C}/\Lambda.$$
Consequently, we have the more general statement
$$\begin{aligned}\Gamma(N)\backslash\mathbb{H}^\pm =&\{\text{complex tori with ordered basis } \alpha : (\mathbb{Z}/N)^2 \xrightarrow{\sim} \tfrac{1}{N}\Lambda/\Lambda\}/\sim\\ =&\{\text{elliptic curves } E \text{ over } \mathbb{C} \text{ with } \alpha : (\mathbb{Z}/N)^2 \xrightarrow{\sim} E(\mathbb{C})[N]\}/\sim .\end{aligned}$$
- In between $\mathrm{GL}_2(\mathbb{Z})$ and $\Gamma(N)$, we had the groups
$$\Gamma_1(N) = \{\gamma \in \mathrm{GL}_2(\mathbb{Z}) | \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1\end{smallmatrix}\right) \bmod N\},$$
$$\Gamma_0(N) = \{\gamma \in \mathrm{GL}_2(\mathbb{Z}) | \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & *\end{smallmatrix}\right) \bmod N\}.$$
- The modular curves associated to these have moduli interpretations:

$$\Gamma_1(N)\backslash\mathbb{H}^\pm =\{\text{elliptic curves } E|\mathbb{C} \text{ with point } Q \in E(\mathbb{C})[N] \text{ of exact order } N\}/\sim$$
$$\Gamma_0(N)\backslash\mathbb{H}^\pm =\{\text{elliptic curves } E|\mathbb{C} \text{ with cyclic subgroup } C \subseteq E(\mathbb{C})[N] \text{ of rank } N\}/\sim .$$

Motivating Example: The Legendre family

- Consider the complex manifold $U = \mathbb{P}^1\backslash\{0, 1, \infty\}$ in the variable λ.
- Over U, we have a complex family of elliptic curves:
- The **Legendre family** is cut out of $\mathbb{P}^2_U \to U$ by the Weierstraß equation
$$\mathbb{E}_\lambda : Y^2 = X(X-1)(X-\lambda).$$

Proposition 7.6. (1) *We have* $\mathbb{E}_\lambda[2] = \{O, (0, 0), (1, 0), (\lambda, 0)\}$. *In particular, there is a natural isomorphism* $\mathbb{E}_\lambda[2] = (\underline{\mathbb{Z}/2\mathbb{Z}})^2$.
(2) *For any elliptic curve E over $\mathbb{C}$ together with a trivialisation*
$$\alpha : (\underline{\mathbb{Z}/2\mathbb{Z}})^2 \xrightarrow{\sim} E[2],$$
there is a unique point $x \in U$ such that $E = (\mathbb{E}_\lambda)_x$.

- **Upshot:** $\mathbb{E}_\lambda \to U$ is **universal elliptic curve** with level $\Gamma(2)$, in some sense.
- This is all algebraic! Can make sense of $\mathbb{E}_\lambda \to U$ as morphism of schemes $\mathbb{C}$.
- Even better: These schemes are already defined over $\mathbb{Q}$.
- Can elaborate on this argument to show that
$$\Gamma(2)\backslash\mathbb{H}^\pm =: X_{\Gamma(2)}(\mathbb{C}) = \mathbb{P}^1\backslash\{0, 1, \infty\}.$$
- NB: In particular, the compactification is $X^*_{\Gamma(2)} = \mathbb{P}^1$.
- This shows that $S_2(\Gamma_0(2)) = 0$, which we used for Fermat's Last Theorem.[11]

Idea:

> Get algebraic model of $\Gamma(N)\backslash\mathbb{H}^\pm$ over $\mathbb{Q}$ by passing from $\mathbb{C}$ to moduli of elliptic curves over general $\mathbb{Q}$-schemes S.

- **Question:** Is there a scheme representing the functor
$$\mathcal{P}_{\mathrm{SL}_2(\mathbb{Z})} : S \mapsto \{\text{elliptic curves over } S\}/\sim$$
on schemes over $\mathbb{Q}$? This would be a scheme $\mathcal{X} \to \mathbb{Q}$ with $\mathbb{C}$-points $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

[11] We used implicitly that $\omega^2 \cong \Omega^1$.

- **Answer:** No. An easy way to see this is the phenomenon of twists:
- Can have two non-isomorphic $E \ncong E'$ over $\mathbb{Q}$ that become isomorphic over a quadratic extension K. In particular, $j(E) = j'(E)$.
- Clearly, two $\mathbb{Q}$-points of $\mathcal{X}$ agree iff they do over K. So $\mathcal{X}$ cannot exist!
- Same problem for the Legendre family above: This does not represent

$$S \mapsto \{\text{elliptic curves } E \text{ over } S \text{ with } \alpha : \underline{(\mathbb{Z}/2\mathbb{Z})}^2 \xrightarrow{\sim} E[2]\}/\sim$$

  because this fails to account for quadratic twists.
- Fact: Twists of elliptic curves E over a field K correspond to 1-cocycles

$$G_{\overline{K}|K} \to \mathrm{Aut}(E).$$

  Quadratic twists come from the involution $([-1] : E \to E) \in \mathrm{Aut}(E)$.
- **Slogan: Nontrivial automorphisms are bad for representability.**[12]
- **Upshot:** In order to get a representable functor, need additional data.

Moduli problems of elliptic curves [Katz–Mazur: Arithmetic moduli...]

- Consider the moduli functors on schemes over $\mathbb{Z}[\frac{1}{N}]$:

$$\mathcal{P}_{\Gamma_0(N)} \colon S \mapsto \{(E|S, C \subseteq E[N] \text{ cyclic subgroup scheme of rank } N)\}/\sim,$$
$$\mathcal{P}_{\Gamma_1(N)} \colon S \mapsto \{(E|S, Q \in E[N](S) \text{ point of exact order } N)/\sim,$$
$$\mathcal{P}_{\Gamma(N)} \colon S \mapsto \{(E|S, \alpha \colon \underline{(\mathbb{Z}/N\mathbb{Z})^2} \xrightarrow{\sim} E[N] \text{ isom of group schemes})/\sim.$$

- We can also form "mixed moduli problems" like

$$\mathcal{P}_{\Gamma_1(N)\cap\Gamma_0(p)} := \mathcal{P}_{\Gamma_1(N)} \times_{\mathcal{P}_{\mathrm{SL}_2(\mathbb{Z})}} \mathcal{P}_{\Gamma_0(p)} \colon S \mapsto \{(E, Q, C)\}/\sim$$

- Crucial point: Let $(E, C) \in \mathcal{P}_{\Gamma_0(N)}(S)$. The automorphism $[-1] : E \to E$ sends $C \mapsto C$. In particular, $[-1] \in \mathrm{Aut}(E, C)$.
- In contrast, let $(E, Q) \in \mathcal{P}_{\Gamma_1(N)}(S)$, then $[-1] : E \to E$ sends $Q \in E[N]$ to $-Q \in E[N]$, which is **different if $\mathbf{N \geq 3}$**. In particular, $[-1] \notin \mathrm{Aut}(E, Q)$.[13]
- Similarly for $\mathcal{P}_{\Gamma(N)}(S)$. This explains the problem with the Legendre family: $N = 2$ is too small to deal with the automorphism $[-1]$! We need $N \geq 3$.
- **Upshot:** The moduli problems $\mathcal{P}_{\Gamma(N)}$ and $\mathcal{P}_{\Gamma_1(N)}$ are "**rigid**" (:= have no non-trivial automorphisms) for $N \geq 3$, in contrast to $\mathcal{P}_{\Gamma_0(N)}$ and $\mathcal{P}_{\Gamma(2)}$.

Theorem. *Let $N \geq 3$ and p any prime.*

(1) *The moduli problems $\mathcal{P}_{\Gamma(N)}$ and $\mathcal{P}_{\Gamma_1(N)}$ are each representable by smooth affine curves $X_{\Gamma(N)}$ and $X_{\Gamma_1(N)}$ over $\mathbb{Z}[\frac{1}{N}]$.*

(2) *For any $n \in \mathbb{N}$, the moduli problem $\mathcal{P}_{\Gamma(N)\cap\Gamma_0(p^n)}$ is representable by a flat affine curve $X_{\Gamma_1(N)\cap\Gamma_0(p^n)}$ over $\mathbb{Z}[\frac{1}{N}]$ that is smooth over $\mathbb{Z}[\frac{1}{pN}]$.*

[12] More precisely, it is the non-flatness (due to exceptional isomorphisms of elliptic curves) of the automorphism group that causes problems. E.g., in many cases the Picard functor parametrizing isomorphism classes of line bundles is representable, but under the assumptions, say, of properness and geometrically integral fibers the automorphism groups of the line bundles are all $\mathbb{G}_m$, and hence flat over the base.

[13] And there are no exceptional isomorphisms.

- Remark: In order to represent $\mathcal{P}_{\mathrm{SL}_2(\mathbb{Z})}$, one can pass from schemes to the bigger category of *stacks*. Get "moduli stack of elliptic curves" $\mathcal{M}_{1,1}$.
- Between varying level structures $\Gamma \subseteq \Gamma' \subseteq \Gamma(N)$, have forgetful morphisms

$$X_{\Gamma'} \to X_\Gamma.$$

These are all finite étale over $\mathbb{Z}[\frac{1}{N}]$.
- $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts from the right on $\mathcal{P}_{\Gamma(N)}$ by precomposition

$$\alpha \mapsto \alpha \circ \gamma.$$

Alternatively, this is a left action via precomposition with $\gamma^\vee = \gamma^{-1} \det \gamma$.
- This induces a $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-action on $X_{\Gamma(N)}$.
- Similarly, there is a natural $(\mathbb{Z}/N\mathbb{Z})^\times$-action on $X_{\Gamma_1(N)}$.
- We can then *define* a modular curve

$$X_{\Gamma_0(N)} := X_{\Gamma_1(N)}/(\mathbb{Z}/N\mathbb{Z})^\times.$$

- This does not represent $\mathcal{P}(\Gamma_0(N))$, but it is "as close as possible".
- In particular, $X_{\Gamma_0(N)}(K) = \mathcal{P}(\Gamma_0(N))(K)$ for algebraically closed K.

Compactification

- The j-invariant associated to Weierstraß equations defines a finite flat function

$$j : X \to \mathbb{A}^1.$$

- By normalisation in $\mathbb{A}^1 \to \mathbb{P}^1$, can define

$$j : X^* \to \mathbb{P}^1,$$

still a finite flat morphism. Think of X^* as X plus a finite divisor of points.
- Fact: X^* is smooth and proper.
- Reason: This can be seen using the Tate curve over $\mathbb{Z}[\frac{1}{N}][[q]]$.
- $X^*(\mathbb{C})$ is the smooth compactification $X^*_{\mathbb{C}}$ of $\Gamma\backslash\mathbb{H}^\pm$ mentioned earlier.
- Remark: X^* is a moduli space of "generalised elliptic curves", cf. [DR73].

Geometric Modular forms References: [Katz], [Loeffler: lectures notes]

- Fix $N \geq 3$. Let $X = X_{\Gamma_1(N)}$.
- Goal: Geometric reinterpretation of modular forms as sections of sheaves on X:

Definition 7.7. *Let* $\omega := e^*\Omega^1_{E|X}$, *where* $e : X \to E$ *is the identity section.*

- Since $E \to X$ is smooth of dimension 1, this is a line bundle.
- Fact: This extends uniquely to a line bundle ω on X^*. Reason: The universal elliptic curve extends to a group scheme $E^* \to X^*$, take $e^*\Omega^1_{E^*|X^*}$.

Proposition 7.8. *The analytification of* $\omega^{\otimes k}$ *is naturally isomorphic to the sheaf* ω^k *of modular forms of weight* k *on* $X^*(\mathbb{C})$.

Sketch of proof. Recall: $\mathbb{H}^\pm$ is moduli space of $E|\mathbb{C}$ with $\alpha : \mathbb{Z}^2 \xrightarrow{\sim} H_1(E,\mathbb{Z})$.
Integration $\int_{\alpha(1,0)}$ defines an isomorphism $e^*\Omega^1_{E|\mathbb{C}} \xrightarrow{\sim} \mathbb{C}$.
Get canonical trivialisation of ω over $\mathbb{H}^\pm$. Check $\mathrm{GL}_2(\mathbb{Z})$-action coincides. □

- In particular, the sheaf ω already has an algebraic model over $\mathbb{Z}[\frac{1}{N}]$.

Definition 7.9. *For any $\mathbb{Z}[\frac{1}{N}]$-algebra R, can define*

$$M_k(\Gamma_1(N), R) = \Gamma(X^*_{\Gamma_1(N)} \times_{\mathrm{Spec}(\mathbb{Z}[1/N])} \mathrm{Spec}(R), \omega^{\otimes k}),$$

the R-module of ***modular forms of weight*** k ***with coefficients in*** R.

- Fact: This is a finite free $\mathbb{Z}[\frac{1}{N}]$-module.
- Fact: Using Tate curves, get q-expansions in $R[[q]]$. $\rightsquigarrow$ Can define cusp forms.
- Fact: If R, S are flat $\mathbb{Z}[\frac{1}{N}]$-algebras (e.g. any $\mathbb{Q}$-algebras), then
$$M_k(\Gamma_1(N), R) \otimes_R S = M_k(\Gamma_1(N), S).$$
- **Upshot:** Modular forms are manifestly objects of algebraic geometry!

8. The Eichler–Shimura relation (by Ben Heuer)

Last time:

- There is a $\mathrm{GL}_2(\mathbb{Z})$-equivariant bijection

$$\begin{array}{ccc} \mathbb{H}^\pm & \to & \{\text{elliptic curves } E \text{ over } \mathbb{C} \text{ with } \alpha\colon \mathbb{Z}^2 \xrightarrow{\cong} H_1(E,\mathbb{Z})\}/\sim \\ \tau & \mapsto & (E_\tau := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau), \alpha\colon \mathbb{Z}^2 \to \mathbb{Z}+\mathbb{Z}\tau,\ (1,0)\mapsto 1, (0,1)\mapsto \tau) \end{array}$$

- Corollary:

$$\mathrm{GL}_2(\mathbb{Z})\backslash(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \times \mathbb{H}^\pm) \cong \{\text{elliptic curves } E \text{ over } \mathbb{C} \text{ with } \alpha\colon \widehat{\mathbb{Z}}^2 \xrightarrow{\sim} TE\}/\sim,$$

  where $TE := \varprojlim_{N\in\mathbb{N}} E[N]$ is the full adélic Tate module of E.
- This can be defined over $\mathbb{Q}$!
- Set $\widehat{X} \to \mathrm{Spec}(\mathbb{Q})$ as the moduli scheme representing the functor

$$\mathbb{Q}\text{–}\mathbf{Sch} \to \mathbf{Sets}, \quad S \mapsto \{\text{elliptic curves } E \text{ over } S \text{ with } \alpha\colon \widehat{\mathbb{Z}}^2 \xrightarrow{\sim} TE\}/\sim.$$

  Here, $TE = \varprojlim_N E[N]$ is an inverse limit of finite, étale group schemes over S.
- $-1 \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$ acts trivially on $\widehat{X}$.
- For $K \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ compact-open with $K \subseteq K_1(3)$ (note in particular, $-1 \notin K$) have smooth quasi-projective curve

$$X_K = \widehat{X}/K.$$

- E.g.: If $K = K(N)$, $N \geq 3$, then

$$X_{K(N)}$$

  parametrizes elliptic curves E together with trivialization $(\mathbb{Z}/N)^2 \xrightarrow{\sim} E[N]$.
- Other examples, $K = K_1(N) \rightsquigarrow \mathbb{Z}/N \hookrightarrow E[N]$, $K = K_0(N) \rightsquigarrow$ subgroups.

Next goal:

- Use modular curves to reinterpret Hecke operators geometrically.
- From geometry of modular curves modulo p, deduce fundamental relation between Galois action and Hecke action: The Eichler–Shimura relation.

The adélic action:

- Clearly, $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ acts on $\widehat{X}$, by precomposition on $\alpha\colon \widehat{\mathbb{Z}}^2 \cong TE$.
- But: Want $\mathrm{GL}_2(\mathbb{A}_f)$-action to get Hecke operators

$\Rightarrow$ work with elliptic curves up to quasi-isogeny.
- Recall: $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and

$$\mathrm{GL}_2(\mathbb{Z})\backslash(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \times \mathbb{H}^\pm) \cong \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm).$$

- On elliptic curves can interpret this as follows.
- Recall that the LHS geometrizes to $\widehat{X}$, which represents the functor

$$P : \mathbb{Q}\text{–}\mathbf{Sch} \to \mathbf{Sets}, \quad S \mapsto \{(E|S, \alpha : \widehat{\mathbb{Z}}^2 \xrightarrow{\sim} TE)\}/\text{isomorphism}$$

- Let $VE = TE \otimes_{\mathbb{Z}} \mathbb{Q}$ be the rational adélic Tate module, and define

$$P' : \mathbb{Q}\text{–}\mathbf{Sch} \to \mathbf{Sets}, \quad S \mapsto \{(E|S, \alpha : \mathbb{A}_f^2 \xrightarrow{\sim} VE)\}/\text{quasi-isogeny}^{14}$$

Definition 8.1. *Recall: An isogeny is a surjective homomorphism $E \to E'$ with finite flat kernel. A **quasi-isogeny** is an element of*

$$\{E \to E' \text{ isogeny}\} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

*A p-**quasi-isogeny** is an element of*

$$\{E \to E' \text{ isogeny of degree a power of } p\} \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{p}].$$

- Then $P \cong P'$. Indeed:
  - Clearly have natural transformation $P \to P'$. Sketch for inverse:
  - Take $(E, \alpha : \mathbb{A}_f^2 \xrightarrow{\sim} VE) \in P'(S)$. Multiply by isogeny $N : E \to E$ until α^{-1} restricts to
  $$TE \to \widehat{\mathbb{Z}}^2.$$
  Reduce both sides mod $M \in \mathbb{N}$. Let D be the kernel of the induced map
  $$E[M] \to (\mathbb{Z}/M)^2,$$
  this stabilises for $M \gg 0$. The dual isogeny $E/D \to E$ induces
  $$\alpha : \widehat{\mathbb{Z}}^2 \xrightarrow{\sim} T(E/D).$$
  - This defines unique representative in $P(S)$. Thus $P(S) \twoheadrightarrow P'(S)$.
- This shows that the natural $\mathrm{GL}_2(\hat{\mathbb{Z}})$-action extends to a $\mathrm{GL}_2(\mathbb{A}_f)$-action.
- This does not fix the projection to any X_K because it changes E.
- Example: $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ acts by sending $E \mapsto E/D$ where $D \subseteq E[p]$ is the subgroup generated by the image of $\alpha(1,0)$ under $T_pE \twoheadrightarrow E[p]$.
- Fact: All of this still works after compactification.

Geometric interpretation of Hecke operators

- Let $V = C^\infty(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm)$ be the smooth $\mathrm{GL}_2(\mathbb{A}_f)$-representation of smooth functions
$$f\colon \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm \to \mathbb{C},\ (g,z) \mapsto f(g,z),$$
i.e., f is fixed under some compact-open subgroup in $\mathrm{GL}_2(\mathbb{A}_f)$ and $z \mapsto f(g,z)$ is smooth for any $g \in \mathrm{GL}_2(\mathbb{A}_f)$.
- Fix a prime p.
- Let $\varphi \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p))$ be any element of the Hecke algebra at p, i.e., a locally constant function $\varphi\colon \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{C}$ with compact support.
- Recall that for $f \in V$ we have
$$\varphi * f(g,z) = \int_{\mathrm{GL}_2(\mathbb{Q}_p)} \varphi(h) f(gh,z)\,dh,$$
where we chose the Haar measure on $\mathrm{GL}_2(\mathbb{Q}_p)$ with volume 1 on $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$.
- The integral above can be rewritten as follows:
- Consider the diagram
$$\begin{array}{ccc} & \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{Q}_p) & \\ {\scriptstyle q_2}\swarrow & & \searrow{\scriptstyle q_1} \\ \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm & & \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm, \end{array}$$

$$q_1(g,z,h) = (g,z),$$
$$q_2(g,z,h) = (gh,z).$$

- Such a diagram is called a **correspondence**.
- We also have the projection to the third factor:

$$\pi\colon \mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}^{\pm}\times\mathrm{GL}_2(\mathbb{Q}_p)\to\mathrm{GL}_2(\mathbb{Q}_p)$$

Lemma 8.2. *For $f : \mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}^{\pm}\to\mathbb{C}$ in V we have*

$$\varphi * f = q_{1,!}(q_2^*(f)\cdot\pi^*(\varphi)),$$

where $(-)^*$ means the pullback of a function and $(-)_!$ integrating along the fiber (which makes sense because φ has compact support).

- *Proof.* The fiber of q_1 over $(g,z)\in\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}^{\pm}$ is

$$q_1^{-1}(g,z) = \{(g,z,h)|\ h\in\mathrm{GL}_2(\mathbb{Q}_p)\}\cong\mathrm{GL}_2(\mathbb{Q}_p)$$

and $q_2^*(f)\cdot\pi^*(\varphi)$ is the function

$$(g,z,h)\mapsto f(gh,z)\varphi(h).$$

Applying $q_{1,!}$, the integral over this fibre, gives

$$\int\limits_{\mathrm{GL}_2(\mathbb{Q}_p)}\varphi(h)f(gh,z)dh = \varphi * f(g,z).$$

Note that this is well-defined because φ has compact support. $\square$

Passage to finite level

- Let N be coprime to p and let $K := K_1(N)\subseteq\mathrm{GL}_2(\mathbb{A}_f)$. Then $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$.
- Write K^p for the prime-to-p-part of K, i.e. $K = K_pK^p$ and $K^p\cap K_p = 1$.
- Assume that f is fixed under K, i.e., arises by pullback from a function

$$\tilde{f}\colon \mathrm{GL}_2(\mathbb{A}_f)/K\times\mathbb{H}^{\pm}\to\mathbb{C}.$$

- Assume $\varphi\in\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p),K_p)$, i.e., that φ is K_p-biinvariant.
- φ is the pullback of a function with finite support

$$\tilde{\varphi}\colon K_p\backslash\mathrm{GL}_2(\mathbb{Q}_p)/K_p\to\mathbb{C}.$$

- In this case, $\varphi * f$ is again K-invariant, i.e., the pullback of a function

$$\widetilde{\varphi * f}\colon \mathrm{GL}_2(\mathbb{A}_f)/K\times\mathbb{H}^{\pm}\to\mathbb{C}.$$

- We can change the above diagram (while retaining notation) to:

$$\begin{array}{ccc} & (\mathrm{GL}_2(\mathbb{A}_f)/K^p\times\mathbb{H}^{\pm})\times^{K_p}\mathrm{GL}_2(\mathbb{Q}_p)/K & \\ & {\scriptstyle q_2}\swarrow \qquad \searrow{\scriptstyle q_1} & \\ \mathrm{GL}_2(\mathbb{A}_f)/K\times\mathbb{H}^{\pm} & & \mathrm{GL}_2(\mathbb{A}_f)/K\times\mathbb{H}^{\pm}. \end{array}$$

Here $-\times^{K_p}-$ is the contracted product from Section 3 (i.e., the quotient for the action $k\cdot(g,z,h) := (gk^{-1},z,kh)$) and

$$q_2([g,z,h]) = [gh,z],\quad q_1([g,z,h]) = [g,z],$$

where square brackets indicate equivalence classes.

- We again have a projection
$$\begin{aligned}\pi\colon (\mathrm{GL}_2(\mathbb{A}_f)/K^p \times \mathbb{H}^\pm) \times^{K_p} \mathrm{GL}_2(\mathbb{Q}_p)/K_p \to & K_p\backslash \mathrm{GL}_2(\mathbb{Q}_p)/K_p,\\ [g,z,h] \mapsto & [h].\end{aligned}$$
- Next, mod out by $\mathrm{GL}_2(\mathbb{Q})$ on the left: Recall this gives a complex manifold
$$X_{\Gamma_1(N)}(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm)$$
- We then need to replace functions f by sections in $\omega^k,\ k \in \mathbb{Z}$, on this.
- More precisely, we obtain

(1)
$$\begin{array}{ccc} & \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K^p \times \mathbb{H}^\pm) \times^{K_p} \mathrm{GL}_2(\mathbb{Q}_p)/K_p & \\ \swarrow^{q_2} & & \searrow^{q_1} \\ X_{\Gamma_1(N)}(\mathbb{C}) & & X_{\Gamma_1(N)}(\mathbb{C}). \end{array}$$

  and for $f \in H^0(X_{\Gamma_1(N)}(\mathbb{C}), \omega^k)$, i.e., a weakly modular form of level $K_1(N)$, we get
$$\varphi * f = q_{1,!}(q_2^*(f) \cdot \pi^*(\varphi)).$$
- To make this precise, use: We have a canonical isomorphism
$$q_2^*(\omega^k) \cong q_1^*(\omega^k)$$
  since ω^k is defined via pullback of a $\mathrm{GL}_2(\mathbb{Q})$-equivariant line bundle on $\mathbb{H}^\pm$.
- Thus can regard $q_2^*(f) \cdot \pi^*(\varphi) \in q_2^*(\omega^k)$ as a section of $q_1^*(\omega^k)$.
- Thus we can sum up along the fibers to obtain the section
$$q_{1,!}(q_2^*(f) \cdot \pi^*(\varphi)) \in \Gamma(X_{\Gamma_1(N)}(\mathbb{C}), \omega^k).$$
- This is a more geometric interpretation of the Hecke action on modular forms.

Interpretation in terms of moduli

- On elliptic curves, (Equation (1)) corresponds to a **Hecke correspondence**:
$$\begin{array}{ccc} & \{(\iota : E_1 \dashrightarrow E_2, \alpha)\} & \\ \swarrow^{q_2} & & \searrow^{q_1} \\ \{(E_2, \alpha')\} & & \{(E_1, \alpha)\} \end{array}$$
  where
  - (E_1, α) is an elliptic curve over $\mathbb{C}$ with a $\Gamma_1(N)$-level structure
$$\alpha : \mathbb{Z}/N\mathbb{Z} \hookrightarrow E_1[N]$$
  - Equivalently, this is the datum of a point $Q := \alpha(1) \in E_1[N](\mathbb{C})$ of exact order N.
  - $(\iota : E_1 \dashrightarrow E_2, \alpha)$ is the data of elliptic curves E_1, E_2 over $\mathbb{C}$ with α a $\Gamma_1(N)$-level structure on E_1, and ι a p-quasi-isogeny:
  - $q_1 : (\iota : E_1 \dashrightarrow E_2, \alpha) \mapsto (E_1, \alpha)$.
  - $q_2 : (\iota : E_1 \dashrightarrow E_2, \alpha) \mapsto (E_2, \alpha')$ where α' is the composition
$$\mathbb{Z}/N\mathbb{Z} \xhookrightarrow{\alpha} E_1[N] \xrightarrow[\sim]{\iota} E_2[N]$$
    (use: ι is an isomorphism on N-torsion as $(p, N) = 1$). Equivalently, this sends Q to the image Q' in $E_2[N]$.

  - The map π sends $(\iota : E_1 \dashrightarrow E_2, \alpha)$ to the class of the isomorphism

$$T_p(\iota)\colon T_pE_1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} T_pE_2 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

    More concretely, choose any bases of T_pE_1 resp. T_pE_2. Then $T_p(\iota)$ is represented by a matrix $A \in \mathrm{GL}_2(\mathbb{Q}_p)$. The class of $A \in K_p\backslash \mathrm{GL}_2(\mathbb{Q}_p)/K_p$ is independent of the chosen basis.

- Problem: Functor of p-quasi-isogenies not representable (only by ind-scheme).
- However: Mainly interested in φ given by the characteristic functions for

$$K_p \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} K_p \quad \leftrightarrow \quad \tilde{S}_p,$$

$$K_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_p \quad \leftrightarrow \quad \tilde{T}_p.$$

- Then term $\varphi(h)$ in the defining integral is supported on this double coset.

$\rightsquigarrow$ Hecke correspondences are supported on subspaces represented by schemes!

- For $\tilde{S}_p$, consider tuples $(\iota : E_1 \dashrightarrow E_2, \alpha)$ of the form

$$([p] : E \to E, \alpha)$$

  i.e. $E_1 = E_2$ and $\iota : E_1 \to E_2$ is multiplication $[p]$.
- For $\tilde{T}_p$, instead need to consider

$$(\iota\colon E_1 \to E_2, \alpha)$$

  where $\iota\colon E_1 \to E_2$ is an isogeny of degree p.
- Equivalently, this is the datum of a cyclic subgroup $\ker \iota \subseteq E_1$ of rank p.
- Last time: These are now representable by schemes, even over $\mathrm{Spec}(\mathbb{Q})$ or suitable $\mathrm{Spec}(\mathbb{Z}[1/N])$!

Hecke action on modular curves – perspective of algebraic geometry

- We now pass to the finite level $K = K_1(N)$ for some $p \nmid N$.
- Corresponding modular curve $X_{\Gamma_1(N)}$ over $\mathbb{Z}[\frac{1}{N}]$ represents pairs (E, Q) over $\mathbb{Z}[\frac{1}{N}]$-schemes S of an elliptic curve E over S with a point

$$Q \in E[N](S).$$

- We start with $\langle p \rangle := \tilde{S}_p$. By the above, just need to multiply Q by p.
- For this, use action of $(\mathbb{Z}/N\mathbb{Z})^\times$ on $X_{\Gamma_1(N)}$.
- By the above, the correspondence for $\tilde{S}_p$ now becomes

$$\begin{array}{ccccc} & & X_{\Gamma_1(N)} & & \\ & \overset{p\cdot}{\underset{\sim}{\swarrow}} & & \overset{\mathrm{id}}{\searrow} & \\ X_{\Gamma_1(N)} & & & & X_{\Gamma_1(N)}. \end{array}$$

- Fact: For any $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, have canonical isomorphism $a^*\omega = \omega$.
- Since the right map is an isomorphism, the integration over fibers is trivial.
- Everything extends to compactifications.
- Putting everything together, we have proved:

  Proposition 8.3. *Consider the operator defined as the composition*

$$M_k(\Gamma_1(N), \mathbb{Z}[\tfrac{1}{N}]) = H^0(X^*_{\Gamma_1(N)}, \omega^k) \xrightarrow{p^*} H^0(X^*_{\Gamma_1(N)}, p^*\omega^k) = M_k(\Gamma_1(N), \mathbb{Z}[\tfrac{1}{N}]).$$

  Then its base-change to $\mathbb{C}$ is the Hecke operator $\tilde{S}_p$.

- For $\tilde{T}_p$, instead of isogenies $(\iota : E \to E', \alpha)$, parametrise triples (E, D, Q), where $D = \ker \iota$ finite cyclic subgroup of rank p. Get maps of moduli functors

$$\begin{aligned} \pi_1 :& (E, D, Q) \mapsto (E, Q), \\ \pi_2 :& (E, D, Q) \mapsto (E/D, Q'), \quad Q' := Q + D = \text{image of } Q \text{ under } E[N] \to (E/D)[N] \end{aligned}$$

- These induce a Hecke correspondence

$$\begin{array}{ccc} & X_{\Gamma_0(p)\cap\Gamma_1(N)} & \\ {}^{\pi_2}\swarrow & & \searrow^{\pi_1} \\ X_{\Gamma_1(N)} & & X_{\Gamma_1(N)}. \end{array}$$

- Fact: π_1 and π_2 extend to compactifications $X^*_{\Gamma_0(p)\cap\Gamma_1(N)} \to X^*_{\Gamma_1(N)}$.
- Fact: π_1 and π_2 are finite flat of degree $p + 1$, even on compactifications.
- Fact: There are canonical isomorphisms $\pi_1^*\omega = \pi_2^*\omega$ (over $\mathrm{Spec}(\mathbb{Q})$)
- Since π_1 has finite fibers, the integral occurring in the definition of the Hecke operator becomes a discrete sum, which we can interpret as a trace map
$$\mathrm{Tr}_\pi : \pi_{1,!}\pi_1^*\omega \to \omega$$
- Combining all this, we have proved:

Proposition 8.4. *The base-change to $\mathbb{C}$ of the operator*
$$H^0(X^*_{\Gamma_1(N)}, \omega^{\otimes k}) \xrightarrow{\pi_2^*} H^0(X^*_{\Gamma_1(N)\cap\Gamma_0(p)}, \omega^{\otimes k}) \xrightarrow{\mathrm{Tr}_{\pi_1}} H^0(X^*_{\Gamma_1(N)}, \omega^{\otimes k}).$$
is the Hecke operator $\tilde{T}_p$.

- Note: This is all defined already over $\mathbb{Q}$ (even over $\mathbb{Z}[\frac{1}{N}]$)! Consequence:

Corollary 8.5. *The eigenvalues of $\tilde{S}_p$, $\tilde{T}_p$ on $M_k(\Gamma_1(N), \mathbb{C})$ are algebraic.*

- To give a more explicit description of $\tilde{T}_p$, we reinterpret this in terms of divisors:
- Let $\mathrm{Div}(X_{\Gamma_1(N)})$ = set of divisors on X ($\approx$ formal sums of points).
- We can then reinterpret $\langle p \rangle := \tilde{S}_p$ and $\tilde{T}_p$ as operators on $\mathrm{Div}(X_{\Gamma_1(N)})$:
$$\begin{aligned} \tilde{S}_p &\colon \mathrm{Div}(X_{\Gamma_1(N)}) \to \mathrm{Div}(X_{\Gamma_1(N)}), \quad [E, Q] \mapsto [E, p \cdot Q], \\ \tilde{T}_p &\colon \mathrm{Div}(X_{\Gamma_1(N)}) \to \mathrm{Div}(X_{\Gamma_1(N)}), \quad [E, Q] \mapsto \sum_{D \subseteq E[p]} [E/D, Q + D], \end{aligned}$$
where E is an elliptic curve, and $Q \in E[N]$ a point of order N.

Modular curves in characteristic p

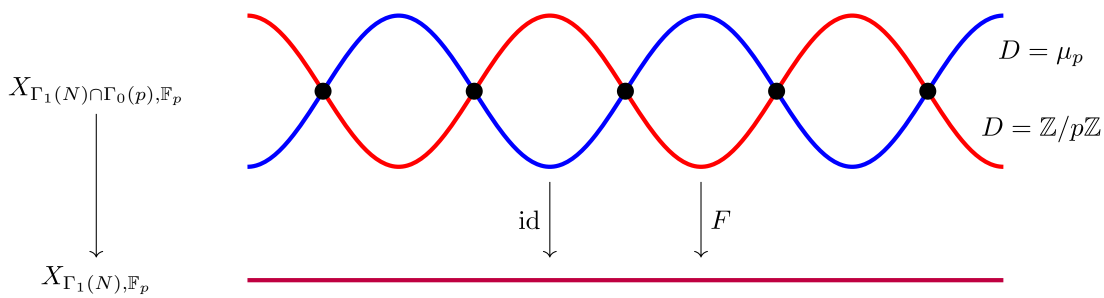
- Let E be an elliptic curve over a scheme S of characteristic p.
- Then $E[N]$ is étale for all $p \nmid N$.
- The morphism $[p] : E \to E$ factors into
$$E \xrightarrow{F} E^{(p)} \xrightarrow{V} E,$$
the Frobenius and Verschiebung isogeny. Here $E^{(p)} = E \times_{S,F} S$.

$\Rightarrow$ always have $\ker F \subseteq E[p]$. This is a connected, i.e. $\ker F(S) = 0$.

Definition 8.6. *Over $S = \overline{\mathbb{F}}_p$, there are two cases:*

(1) *$E(\overline{\mathbb{F}}_p) = \mathbb{Z}/p\mathbb{Z}$. Then E is called ordinary, and $E[p] = \mu_p \times \mathbb{Z}/p\mathbb{Z}$.*
(2) *$E(\overline{\mathbb{F}}_p) = 0$. Then E is called supersingular, and $E[p]$ is connected.*

- "supersingular" is *not* about smoothness. It just means "really special":
- Fact: There are only finitely many isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}}_p$. All others are ordinary.
- Recall: $X_{\Gamma_1(N)}$ is defined over $\mathbb{Z}[\frac{1}{N}]$.
- Can therefore reduce to $\mathbb{F}_p$ to get modular curve $X_{\Gamma_1(N),\mathbb{F}_p} \to \mathrm{Spec}(\mathbb{F}_p)$.
- Similarly for $X_{\Gamma_1(N)\cap\Gamma_0(p)}$ representing (E, D, Q) with $D \subseteq E[p]$ of rank p.
- Deligne–Rapoport: The fibre $X_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{F}_p} \to \mathrm{Spec}(\mathbb{F}_p)$ is of the form



- Two copies of $X_{\Gamma_1(N),\mathbb{F}_p}$, corresponding to $D = \mu_p$ or $D = \mathbb{Z}/p\mathbb{Z}$ as subgroup of $E[p]$, with transversal intersections at supersingular points (black).
- Projection is identity (deg=1) on one copy, and Frobenius (deg=p) on other.

Eichler–Shimura relation [Diamond–Shurman], [Conrad: Appendix to Serre's...]

- Eichler–Shimura relation expresses reduction $\tilde{T}_p$ mod p in terms of Frobenius:
- Base change to algebraic closure $\overline{\mathbb{F}}_p$.
- Consider morphism $\mathrm{Div}^0(X^*_{\Gamma_1(N),\overline{\mathbb{F}}_p}) \to \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{F}}_p})$
- Fact: $\mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{F}}_p})$ is points of a group scheme over $\overline{\mathbb{F}}_p$. In particular, multiplication by p factors through Frobenius F: there is $V = F^\vee$ s.t.

$$p = V \circ F = F \circ V$$

Theorem. *(Eichler–Shimura relation) In $\mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{F}}_p})$, we have*

$$\tilde{T}_p = F + \langle p\rangle V.$$

- Note: There is a version for $X^*_{\Gamma_0(N),\overline{\mathbb{F}}_p}$, which famously reads

$$\boxed{\tilde{T}_p = F + V}$$

We get a second important variant by multiplying by F and using $FF^\vee = p$.

$$F^2 - \tilde{T}_p F + \langle p\rangle p = 0.$$

Proof. (Sketch) Let E be an ordinary elliptic curve over $\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$. Let $C \subseteq E[p]$ be the "canonical subgroup":= generated by kernel of

$$E[p](\breve{\mathbb{Z}}_p) \to E[p](\overline{\mathbb{F}}_p).$$

- Key fact: For $D \subseteq E[p]$ cyclic subgroup scheme of rank p,
  (1) the isogeny $E \to E/D$ reduces to $F : \overline{E} \to \overline{E}^{(p)} \Leftrightarrow D = C$.
  (2) the isogeny $E \to E/D$ reduces to $V : \overline{E} \to \overline{E}^{(p^{-1})} \Leftrightarrow D \neq C$.
- Here $\overline{E}^{(p^{-1})} := E \times_{\overline{\mathbb{F}}_p, F^{-1}} \overline{\mathbb{F}}_p$ = base-change along inverse of Frobenius.
- F sends $Q \in E[N]$ to base-change $Q^{(p)} \in E^{(p)}[N] = E[N]^{(p)}$.
- Since $V \circ F = p$ on $E[N]$, this implies: V sends Q to $pQ^{(p^{-1})} \in E^{(p^{-1})}[N]$.
- Recall: There are $p+1$ different subgroups D of $E[p]$ over $W(\overline{\mathbb{F}}_p)$.
- Thus $\tilde{T}_p([E,Q]) = \sum_{D \subseteq E[p]} [E/D, Q+D]$ in $\mathrm{Div}(X_{\Gamma_1(N)})$ reduces to

$$[E^{(p)}, Q^{(p)}] + p[E^{(p^{-1})}, pQ^{(p^{-1})}] \quad \text{in } \mathrm{Div}(X_{\Gamma_1(N), \overline{\mathbb{F}}_p}).$$

- The first summand is $F[E,Q]$.
- By definition, $\langle p \rangle [E^{(p^{-1})}, Q^{(p^{-1})}] = [E^{(p^{-1})}, pQ^{(p^{-1})}]$.
- Now pass to $\mathrm{Pic}^0(X^*_{\Gamma_1(N), \overline{\mathbb{F}}_p})$. Here: $p = VF$. We therefore have

$$p[E^{(p^{-1})}, Q^{(p^{-1})}] = pF^{-1}[E,Q] = V[E,Q].$$

  (technically, we need to work with $[E,Q] - [E', Q']$ to be in Div^0). $\square$

9. Galois representations associated to newforms

Next aim:

- Realize newforms in cohomology.
- Finish construction of Galois representations associated to newforms of weight $k = 2$.
- Give hints on how to proceed in the case that $k \geq 2$.

We will associate classes in singular cohomology associated to modular forms (of weight 2) by using differential forms. For this we need to recall some facts on de Rham cohomology.

De Rham cohomology (cf. [BT13]**):**

- Let X be a real manifold.
- Let $C^\infty(X)$ be the space of smooth functions $f\colon X \to \mathbb{C}$.
- Let $\mathcal{A}^i(X)$ be the space of smooth i-forms on X, $i \geq 0$.
- The de Rham complex of X is the complex

$$\mathcal{A}^\bullet(X)\colon \quad \underbrace{\mathcal{A}^0(X)}_{=C^\infty(X)} \xrightarrow{d} \mathcal{A}^1(X) \xrightarrow{d} \mathcal{A}^2(X) \to \dots$$

  with d the exterior derivative.
- The de Rham cohomology of X is then defined as the cohomology of the de Rham complex

$$H^*_{\mathrm{dR}}(X) := H^*(\mathcal{A}^\bullet(X))$$

- Fascinatingly, the de Rham comparison isomorphism implies

$$H^*_{\mathrm{dR}}(X) \cong H^*(X, \underline{\mathbb{C}})$$

  with RHS the sheaf cohomology of the constant sheaf $\underline{\mathbb{C}}$ on X. This has the interesting consequence that the de Rham cohomology is a purely topological invariant of X.
  - Sketch of proof: Sending $U \subseteq X$ open to $\mathcal{A}^\bullet(U)$ defines a complex of sheaves

$$\mathcal{A}^\bullet_X$$

    on X which is a resolution of the constant sheaf $\underline{\mathbb{C}}$ (by the Poincaré lemma). Using a partition of unity argument (needs X paracompact) the sheaves $\mathcal{A}^k_X$, $k \geq 0$, are flasque. $\Rightarrow R\Gamma(X, \underline{\mathbb{C}}) \cong \Gamma(X, \mathcal{A}^\bullet_X)$.
- Moreover,

$$H^*(X, \underline{\mathbb{C}}) \cong H^*_{\mathrm{sing}}(X, \mathbb{C}).$$

  with RHS = singular cohomology of X with values in $\mathbb{C}$.
- If Γ is a discrete group acting properly discontinuously and freely on X, then

$$\mathcal{A}^\bullet(\Gamma\backslash X) \cong \mathcal{A}^\bullet(X)^\Gamma = (\underbrace{\mathcal{A}^0(X)^\Gamma}_{=C^\infty(X)^\Gamma} \xrightarrow{d} \mathcal{A}^1(X)^\Gamma \xrightarrow{d} \mathcal{A}^2(X)^\Gamma \to \dots),$$

  i.e., the invariants in the de Rham complex calculate the cohomology of the quotient $\Gamma\backslash X$. Note that due to the potential presence of group cohomology the statement does not hold on cohomology groups.

- Finally, we set

$$H^*_{\mathrm{dR}}(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}^{\pm})) := \varinjlim_K H^*_{\mathrm{dR}}(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K\times\mathbb{H}^{\pm}))$$

  and

$$H^*(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}^{\pm}),\underline{\mathbb{C}}) := \varinjlim_K H^*(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K\times\mathbb{H}^{\pm}),\underline{\mathbb{C}}),$$

  where the colimit is over all compact-open subgroups $K\subseteq \mathrm{GL}_2(\mathbb{A}_f)$, and the transition maps are induced by pullback.
- Then, for $i\geq 0$, the ($\mathrm{GL}_2(\mathbb{A}_f)$-equivariant) de Rham comparison

$$H^i_{\mathrm{dR}}(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}^{\pm}))\cong H^i(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}^{\pm}),\underline{\mathbb{C}})$$

  holds by passing to the colimit.

Upshot:

- Can construct classes in $H^*(X,\underline{\mathbb{C}})$ using (closed) differential forms.

Cohomology classes associated to modular forms of weight 2:

- Note that

$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}^{\pm})\cong \mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}),$$

  where $\mathrm{GL}_2(\mathbb{Q})^+\subseteq\mathrm{GL}_2(\mathbb{Q})$ is the subgroup of elements of positive determinant.
- Pick $f\in H^0(\mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}),\omega^{\otimes k}), k\in\mathbb{Z}$.
- We view f as a function

$$f\colon \mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}\to\mathbb{C},\ (g,z)\mapsto f(g,z)$$

  satisfying

$$f(\gamma g,\gamma z)=(cz+d)^k f(g,z)$$

  for $\gamma\in\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Q})^+$.
- Assume that f is of weight $k=2$.
- Then the differential form $f(g,z)dz$ on $\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}$ is closed and satisfies

$$\gamma^*(f(g,z)dz)=f(\gamma g,\gamma z)\gamma^* dz=\det(\gamma)f(g,z)dz$$

  for $\gamma\in\mathrm{GL}_2(\mathbb{Q})^+$.
- Indeed:
  - Closedness follows from holomorphicity as

$$d(f(g,z)dz)=\frac{\partial}{\partial z}f(g,z)dz\wedge dz-\frac{\partial}{\partial\overline{z}}f(z)dz\wedge d\overline{z}=0.$$

  - If $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Q})$, then

$$\gamma^*(dz)=\frac{\det(\gamma)}{(cz+d)^2}dz.$$

  - Using that f is of weight 2, we can conclude.

- Let
$$|-| = |-|_{\text{adélic}} \colon \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{R}_{>0}$$
be the adélic norm, i.e.,
$$|(x_2, x_3, \ldots, x_\infty)| := \prod_p |x_p|_p \cdot |x_\infty|_\infty$$
with $|-|_p \colon \mathbb{Q}_p \to \mathbb{R}_{\geq 0},\ x \mapsto p^{-v_p(x)}$ the p-adic norm, and $|-|_\infty \colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ the real norm.
- $|-|$ defines the character
$$\chi_{\text{no}} := |-| \circ \det = |\det|_{\text{adélic}}$$
of $\mathrm{GL}_2(\mathbb{A})$, which is trivial on $\mathrm{GL}_2(\mathbb{Q})$.
- Thus, for
$$\tilde{f}(g, z) := \chi_{\text{no}}((g,1))f(g,z)$$
(with $(g,1) \in \mathrm{GL}_2(\mathbb{A}) \cong \mathrm{GL}_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{R})$), the form
$$\eta_f := \tilde{f}(g,z)dz$$
on $\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}$ is $\mathrm{GL}_2(\mathbb{Q})^+$-equivariant.
- We get a map, *which is not* $\mathrm{GL}_2(\mathbb{A}_f)$*-equivariant*,
$$\alpha \colon M_2 \to H^1_{\mathrm{dR}}(\mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H})) \cong H^1(\mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}), \underline{\mathbb{C}})$$
by sending f to $[\eta_f]$.
- For a smooth $\mathrm{GL}_2(\mathbb{A}_f)$-representation V and a smooth character
$$\chi \colon \mathrm{GL}_2(\mathbb{A}_f) \to \mathbb{C}^\times$$
define $V(\chi) := V \otimes_{\mathbb{C}} \chi$.
- Then
$$\alpha \colon M_2(\chi_{\text{no}}) \to H^1(\mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}), \underline{\mathbb{C}})$$
is $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant.
- Let $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ be compact-open, and set
$$X_K := \mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}).$$
with canonical compactification
$$X_K^* := \mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^*)$$
where $\mathbb{H}^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ (equipped with Satake topology).
- If $f \in S_2(K)$, then η_f extends to a holomorphic differential form on X_K^*.
  - Indeed: If $q = e^{2\pi i z}$, then
$$dq = 2\pi i q dz,$$
i.e.,
$$dz = \frac{1}{2\pi i q} dq.$$
Now express $f(g,z)$ at each cusp in Fourier expansion, i.e., as a function of q.

- The diagram
$$\begin{array}{ccc} H^1_{\mathrm{dR}}(X_K^*) & \longrightarrow & H^1_{\mathrm{dR}}(X_K) \\ \downarrow \simeq & & \downarrow \simeq \\ H^1(X_K^*, \underline{\mathbb{C}}) & \longrightarrow & H^1(X_K, \underline{\mathbb{C}}) \end{array}$$
commutes.
- Moreover, the morphism
$$H^1_c(X_K, \underline{\mathbb{C}}) \to H^1(X_K^*, \underline{\mathbb{C}}) = H^1_c(X_K^*, \underline{\mathbb{C}})$$
is surjective as its cokernel embeds into $H^1(\{\text{cusps}\}, \underline{\mathbb{C}}) = 0$.
- Thus, $\alpha(S_2(K))$ lies in the *interior cohomology* $\widetilde{H}^1(X_K, \underline{\mathbb{C}}) \cong H^1(X_K^*, \underline{\mathbb{C}})$ of X_K, which by definition is the image of the compactly supported cohomology $H^1_c(X_K, \underline{\mathbb{C}}) \to H^1(X_K, \underline{\mathbb{C}})$.
- Note that the above discussion applies similarly to *anti*holomorphic modular forms, i.e., complex conjugates of modular forms.
- This yields the map
$$\overline{\alpha}\colon \overline{S_2(K)} \to \widetilde{H}^1(X_K, \underline{\mathbb{C}}),\ \overline{f} \mapsto [\chi_{\mathrm{no}} \overline{f} d\overline{z}],$$
where $\overline{S_2(K)}$ denotes the $\mathbb{C}$-linear space of antiholomorphic modular forms.
- We can pass to infinite level and obtain the morphism
$$\alpha \oplus \overline{\alpha}\colon S_2 \oplus \overline{S_2} \to \widetilde{H}^1(\mathrm{GL}_2(\mathbb{Q}) \backslash (\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}), \underline{\mathbb{C}})$$
with (hopefully) self-explaining notation.

Theorem 9.1 (Eichler–Shimura)**.** *The map*
$$\alpha \oplus \overline{\alpha}\colon S_2(\chi_{no}) \oplus \overline{S_2}(\chi_{no}) \to \widetilde{H}^1(\mathrm{GL}_2(\mathbb{Q}) \backslash (\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}), \underline{\mathbb{C}})$$
is a $\mathrm{GL}_2(\mathbb{A}_f)$*-equivariant isomorphism.*

- The statement is equivalent to the analogous statement for all (sufficiently small) compact-open subgroups $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$.
- Thus, fix some $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ compact-open.
- The proof will exploit the $\cup$-product pairing
$$H^1_c(X_K, \underline{\mathbb{C}}) \times H^1(X_K, \underline{\mathbb{C}}) \to H^2_c(X_K, \underline{\mathbb{C}}) \xrightarrow{\text{integrate}} \mathbb{C},$$
which under the de Rham comparison is induced from the pairing
$$(\eta_1, \eta_2) \mapsto \int\limits_{X_K} \eta_1 \wedge \eta_2$$
of differential forms, where η_1 has compact support (cf. [BT13]).
- We will use the following observation:
  - We can calculate the compactly supported cohomology via $H^*_c(X_K, \underline{\mathbb{C}}) \cong H^*(\mathcal{A}^\bullet_c(X_K))$, where $\mathcal{A}^\bullet_c$ denotes differential forms with compact supports.
  - The canonical map
$$\mathcal{A}^\bullet_c(X_K) \to \mathcal{A}^\bullet_{\mathrm{rd}}(X_K) \tag{2}$$
is a quasi-isomorphism, where the RHS denotes differentials forms of *rapid decay*, cf. [Bor80, Theorem 2].
- For $f \in S_2$ the form η_f is rapidly decreasing.

  - Use that $|e^{2\pi i n z}| = e^{-2\pi n \mathrm{Im}(z)}$ decreases rapidly if $\mathrm{Im}(z) \to \infty$ and $n \geq 1$.
- Thus α lifts naturally to a map

$$\widetilde{\alpha}\colon S_2(K) \to H^1_c(X_K, \underline{\mathbb{C}}).$$

- Consider $f \in S_2(K), g \in \overline{S_2(K)}$. Then:
  * $\widetilde{\alpha}(f) \cup \alpha(f) = 0$ as $dz \wedge dz = 0$.
  * Similarly for g.
  * Using Stokes' theorem, one proves

$$\mathrm{tr}(\widetilde{\alpha}(f) \cup \overline{\alpha}(g)) = \int_{X_K} \eta_f \wedge \eta_g.$$

  * This implies that the Poincaré pairing induces (up to a scalar) the Petterson scalar product aka L^2-pairing $\langle -, - \rangle$ on S_2.
- We can now prove that $\beta := \alpha \oplus \overline{\alpha}$ is injective.
- Indeed:
  - If $\beta(f + \overline{g}) = 0$, then

$$\beta(f + \overline{g}) \cup \overline{\alpha}(\overline{f}) = \langle f, f \rangle = 0,$$

    i.e., $f = 0$. Similarly, $\overline{g} = 0$.
- Now we check $\widetilde{H}^1(X_K, \underline{\mathbb{C}}) = 2\mathrm{dim}(S_2(K))$. If we succeed, the proof of the Eichler-Shimura isomorphism is finished.
- Namely (we use lower case latters to denote dimensions):
  * $\widetilde{h}^1(X_K, \underline{\mathbb{C}})) = h^1(X_K^*, \underline{\mathbb{C}}) = 2 \cdot h^0(X_K^*, \underline{\mathbb{C}}) - \chi_{\mathrm{top}}(X_K^*)$
  * Here: χ_{top} is the *topological* Euler characteristic.
  * On the other hand [15]:

$$\begin{array}{cc} & \dim(S_2(K)) \\ = & h^0(X_K^*, \Omega^1_{X_K^*}) \\ \overset{\mathrm{RR}}{=} & \deg(\Omega^1_{X_K^*}) + \chi_{\mathrm{hol}}(X_K^*) + h^1(X_K^*, \Omega^1_{X_K^*}) \\ = & -\chi_{\mathrm{hol}}(X_K^*) + h^0(X_K^*, \underline{\mathbb{C}}) \end{array}$$

  * Here: χ_{hol} is the *holomorphic* Euler characteristic.
  * $\chi_{\mathrm{top}}(X_K^*) = 2\chi_{\mathrm{hol}}(X_K^*)$

We can now finish our outlined strategy for associating Galois representations fo newforms (at least in weight 2).

Galois representations associated to newforms:

- Recall that in the last two lectures we introduced a scheme

$$\widehat{X} \to \mathrm{Spec}(\mathbb{Q})$$

  with $\mathrm{GL}_2(\mathbb{A}_f)$-action such that naturally

$$\widehat{X}(\mathbb{C}) \cong \mathrm{GL}_2(\mathbb{Q}) \backslash (\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}).$$

- For $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ compact-open (plus sufficiently small), we get a quasi-projective, smooth curve

$$X_K \to \mathrm{Spec}(\mathbb{Q})$$

  with

$$X_K(\mathbb{C}) \cong \mathrm{GL}_2(\mathbb{Q}) \backslash (\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^{\pm})$$

[15] We use implicitly that $\omega^2(-\mathrm{cusps}) \cong \Omega^1$.

(note the clash in notation with previous section, where X_K was equal to RHS).
- Fix a prime ℓ.
- For $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ can consider the interior étale cohomology

$$\widetilde{H}^1_{\text{ét}}(X_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) := \mathrm{Im}(H^1_{c,\text{ét}}(X_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) \to H^1_{\text{ét}}(X_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)) \cong H^1_{\text{ét}}(X^*_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell).$$

- In the limit, we get the big cohomology space

$$\widetilde{H}^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) := \varinjlim_K \widetilde{H}^1_{\text{ét}}(X_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell).$$

- By naturality of $H^1_{c,\text{ét}}, H^1_{\text{ét}}$ this space has an action of

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A}_f),$$

i.e., an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and an action of $\mathrm{GL}_2(\mathbb{A}_f)$, and these two actions commute.
- Fix an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$.
- Using étale comparison theorems (and ι) we get the isomorphisms

$$\widetilde{H}^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) \cong H^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{C}}}, \overline{\mathbb{Q}}_\ell) \cong \widetilde{H}^1(\widehat{X}(\mathbb{C}), \overline{\mathbb{Q}}_\ell) \overset{\iota}{\cong} \widetilde{H}^1(\widehat{X}(\mathbb{C}), \underline{\mathbb{C}}).$$

- These isomorphisms are $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant.
- Let $\pi \subseteq S_2$ be an irreducible $\mathrm{GL}_2(\mathbb{A}_f)$-representation.
- Using ι, we will view each smooth $\mathrm{GL}_2(\mathbb{A}_f)$-representation over $\mathbb{C}$ as a smooth representation over $\overline{\mathbb{Q}}_\ell$.
- The space

$$\widetilde{\rho}_\pi := \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A}_f)}(\pi(\chi_{\text{no}}), \widetilde{H}^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell))$$

is two-dimensional by the Eichler-Shimura isomorphism

$$\widetilde{H}^1(\widehat{X}(\mathbb{C}), \underline{\mathbb{C}}) \cong S_2(\chi_{\text{no}}) \oplus \overline{S_2(\chi_{\text{no}})}$$

and because S_2 is multiplicity free as a $\mathrm{GL}_2(\mathbb{A}_f)$-representation.
- In summary, as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A}_f)$-representation

$$\widetilde{H}^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_\pi \widetilde{\rho}_\pi \otimes_{\overline{\mathbb{Q}}_\ell} \pi(\chi_{\text{no}}), \tag{3}$$

where the sum is running over all irreducible $\mathrm{GL}_2(\mathbb{A}_f)$-representations $\pi \subseteq S_2$ (note that there is a natural morphism from the RHS to the LHS).

As we tried to explain the importance of the Langlands reciprocity for newforms lies in the fact that Hecke eigenvalues match with traces of Frobenii. We will discuss this now.

The Eichler–Shimura relation in étale cohomology:

⚠ We need to show that Hecke eigenvalues match with traces of Frobenii. ⚠
- Recall: To $\pi \subseteq S_k$ with system of Hecke eigenvalues

$$\{a_p(\pi), b_p(\pi) = \chi(p)p^{k-1}\}_{p \notin S},$$

we want to attach a 2-dimensional ℓ-adic representation

$$\rho_\pi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell),$$

such that for $p \notin S$ each *arithmetic* Frobenius $\mathrm{Frob}_p^{\text{arith}}$ at p has characteristic polynomial

$$X^2 - a_p(\pi)X + \chi(p)p^{k-1}$$

(we omit ι, ρ_π in the following), i.e.,

$$(\mathrm{Frob}_p^{\mathrm{arith}})^2 - a_p(\pi)\mathrm{Frob}_p^{\mathrm{arith}} + \chi(p)p^{k-1} = 0.$$

- By passing to $K_1(N)$-invariants in (Equation (3)), we have (for $N \geq 3$)

$$H^1_{\text{ét}}(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_\pi \widetilde{\rho}_\pi \otimes_{\overline{\mathbb{Q}}_\ell} \pi(\chi_{\mathrm{no}})^{K_1(N)}$$

(note $X_{K_1(N)} \cong X_{\Gamma_1(N)}$).
- Let p be a prime, $p \nmid \ell N$.
- Fix a place of $\overline{\mathbb{Q}}$ over p. This determines an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$.
- Recall: For an abelian variety A over a field L, and ℓ a prime, we write

$$T_\ell A = \varprojlim A[\ell^n](\overline{L}), \quad V_\ell A = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

for the ℓ-adic Tate module resp. the rationalized ℓ-adic Tate module.

Lemma 9.2. *Let p be prime, $p \nmid \ell N$. Then we have*

$$H^1_{\text{ét}}(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{F}}_p})^\vee,$$

where $(-)^\vee$ denotes the $\mathbb{Q}_\ell$-dual.

Proof. The Kummer sequence $1 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{\ell^n} \mathbb{G}_m \to 1$ implies

$$H^1_{\text{ét}}(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1)) \cong V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}).$$

The Weil pairing yields a canonical isomorphism

$$V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}})^\vee \cong V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}})(-1).$$

Finally, because $\ell \neq p$

$$V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}) \cong V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}_p}) \cong V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{F}}_p})$$

by lifting torsion points. □

- In particular, the Galois representation

$$H^1_{\text{ét}}(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$$

is unramified at p.
- On $V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{F}}_p})$, we have the Eichler–Shimura relation, which we recall reads

$$F^2 - \widetilde{T}_p F + p\widetilde{S}_p = 0$$

with F the arithmetic Frobenius.

Proposition 9.3. *Let p be a prime with $p \nmid \ell N$. Then on $H^1_{\text{ét}}(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$, we have*

$$(\mathrm{Frob}_p^{\mathrm{geom}})^2 - \widetilde{T}_p \mathrm{Frob}_p^{\mathrm{geom}} + p\widetilde{S}_p = 0.$$

- For a normalized newform $f \in S_2(\Gamma_0(N), \chi)$ we define finally

$$\rho_f := (\widetilde{\rho}_\pi)^\vee$$

with π the irreducible $\mathrm{GL}_2(\mathbb{A}_f)$-representation generated by f, and

$$\widetilde{\rho}_\pi := \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A}_f)}(\pi(\chi_{\mathrm{no}}), \widetilde{H}^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)).$$

Some analysis of ρ_f, cf. [Rib77]**:**

- Write the (normalized) newform $f \in S_2(\Gamma_0(N), \chi)$ in Fourier expansion
$$f(q) = \sum_{n \geq 1} a_n q^n.$$
Then for $p \nmid \ell N$, $\mathrm{Frob}_p^{\mathrm{arith}}$ has characteristic polynomial
$$X^2 - a_p X + \chi(p)p$$
on ρ_f.
- Indeed:
  - By the (dual of the) Eichler–Shimura relation for $\mathrm{Frob}_p^{\mathrm{geom}}$ we get
  $$(\mathrm{Frob}_p^{\mathrm{arith}})^2 - \widetilde{T}_p \mathrm{Frob}_p^{\mathrm{arith}} + p\widetilde{S}_p$$
  on ρ_f.
  - Recall that $\widetilde{T}_p$ has eigenvalue pa_p on $\pi^{K_1(N)}$, while $\widetilde{S}_p$ has eigenvalue $\chi(p)p^2$.
  - We had to twist the $\mathrm{GL}_2(\mathbb{A}_f)$-action on S_2 by $\chi_{\mathrm{no}} = |\det(-)|$. Thus $\widetilde{T}_p$ has eigenvalue a_p on
  $$\widetilde{\rho}_\pi \otimes_{\overline{\mathbb{Q}}_\ell} \pi(\chi_{\mathrm{no}})^{K_1(N)} \subseteq H^1_{\text{ét}}(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell).$$
  while $\widetilde{S}_p$ has eigenvalue $\frac{1}{p^2}\chi(p)p^2 = \chi(p)$.
- The determinant of ρ_f is
$$\psi \cdot \chi_{\mathrm{cyc}},$$
where
$$\chi_{\mathrm{cyc}} \colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell^\times$$
denotes the cyclotomic character, and
$$\psi \colon \mathbb{Q}^\times \mathbb{R}_{>0} \backslash \mathbb{A}^\times \to \overline{\mathbb{Q}}_\ell^\times \overset{\iota}{\cong} \mathbb{C}^\times$$
the (finite order) adélic character determined by χ.
- Indeed:
  - One uses Chebotarev and checks that both sides agree on arithmetic Frobenius elements for $p \nmid \ell N$.
- In particular, ρ_f is *odd*, i.e., the determinant of each complex conjugation is -1.
  - This uses that for all $k \geq 1$ non-triviality of $S_k(\Gamma_0(N), \chi)$ implies $\chi(-1) = (-1)^k$.
- ρ_f is irreducible, cf. [Rib77, Theorem (2.3)].
- Because, ρ_f is realized in the étale cohomology of a proper, smooth scheme over $\mathbb{Q}$, the de Rham comparison theorem in ℓ-adic Hodge theory implies that $\rho_{f|G_{\mathbb{Q}_\ell}}$ is de Rham.

Outline of the construction in weights ≥ 2:

- Instead of
$$\widetilde{H}^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$$
use
$$\widetilde{H}^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{Q}}}, \mathbb{L}^k),$$
with the local system
$$\mathbb{L}^k := \mathrm{Sym}^{k-1} R^1 f_*(\overline{\mathbb{Q}}_\ell)$$

for $f\colon E \to \widehat{X}$ the universal elliptic curve.

- Problem: $\mathbb{L}^k$ does not extend to a local system on the compactification. Thus one needs an additional argument to justify base change to $\overline{\mathbb{F}}_p$.
- Use similar strategy to obtain analogs of the Eichler–Shimura relation/isomorphism.
- Also replace the de Rham comparison/étale comparsion theorems by their versions with coefficients.
- More details can be found in [Del71b].

10. Galois representations for weight 1 forms (by Ben Heuer)

Last time: Constructed Galois representations attached to newforms in weight ≥ 2:

Theorem 10.1 (Deligne '71). *Let $N \geq 3$ and $k \geq 2$. Let $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Let $f \in M_k(\Gamma_0(N), \varepsilon)$ be a newform of weight k. Let K be the number field generated by the $a_p(f)$ and ε. Then for any place λ of K over a prime $\ell \nmid N$, there is a continuous, odd, irreducible Galois representation*

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_\lambda),$$

unramified outside N, such that the characteristic polynomial of Frob_p is

$$X^2 - a_p X + \varepsilon(p)p^{k-1} \quad \text{for all } p \nmid N\ell.$$

Galois reps associated to weight 1 modular forms

- Goal of today: prove the following Theorem

Theorem 10.2 (Deligne–Serre '73). *Let $N \geq 3$. Let $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Let $f \in M_1(\Gamma_0(N), \varepsilon)$ be a newform of weight 1. Then there is a continuous, odd, irreducible Galois representation*

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C}),$$

unramified outside N, such that the characteristic polynomial of Frob_p is

$$X^2 - a_p X + \varepsilon(p) \quad \text{for all } p \nmid N.$$

- This finishes the construction of newforms $\rightsquigarrow$ Galois reps.
- Note that ρ has image in $\mathrm{GL}_2(\mathbb{C})$, i.e. is an Artin representation, in contrast to the ones for weight ≥ 2, which were ℓ-adic. Why is that?
  - Easy fact: Any Artin representation has finite image.
  - In fact, ρ is defined already over a number field, so could embed into p-adic field K_λ as before. But using $\mathbb{C}$ is a way of signaling "finite image".
  - On the Galois side, see that ρ can only be finite for $k = 1$ since the determinant is required to be $\det \rho = \varepsilon\chi_{\mathrm{cycl}}^{k-1}$, and χ_{cycl} has infinite image.
- If ε is even, then $M_1(\Gamma_0(N), \varepsilon) = 0$. Hence ε is odd, which implies ρ odd.

First sketch of construction

Compared to last time, the proof will be less geometry, more Galois representation theoretic:

- Reduce f mod p. This gives a "mod p modular form"
- Use congruences of modular forms to show there is a mod p Hecke eigenform f' with same eigenvalues but higher weight ≥ 2.
- Lift f' back to characteristic 0 using "Deligne–Serre lifting"
- Attach Galois representation $\rho_{f'}$ using weight ≥ 2 construction
- Reduce Galois representation mod ℓ
- Lift Galois representation from $\mathbb{F}_\ell$ to $\mathbb{C}$.

Introduction to mod p modular forms

- Let $N \geq 3$. Let S be any $\mathbb{Z}[\frac{1}{N}]$-scheme S.
- Recall: We had defined modular forms over S of weight k by
$$M_k(\Gamma_1(N); S) := \Gamma(X^*_{\Gamma_1(N),S}, \omega^k).$$
- ω is the ample line bundle of relative differentials on universal elliptic curve
- Can define q-expansions: Let $S = \mathrm{Spec}(A)$, then have the Tate curve $T(q)$ over $\mathbb{Z}((q)) \otimes_{\mathbb{Z}} A$. Over $\mathrm{Spec}(\mathbb{Z}((q)) \otimes_{\mathbb{Z}} A)$, have canonical isomorphism
$$\omega_{\mathbb{G}_m} = \omega_{T(q)}$$
- bundle $\omega_{\mathbb{G}_m}$ is trivial: the canonical differential $\frac{dx}{x}$ on $\mathbb{G}_m$ is invertible section
- Via this trivialisation, get for every cusp of $X^*_{\Gamma_1(N),A}$ an associated q-expansion
$$M_k(\Gamma_1(N); S) \to A[[q]].$$

Theorem 10.3 (Katz' q-expansion principle)**.** *This is injective for all k at each cusp.*

- Now apply this all to $S = \mathrm{Spec}(\mathbb{F})$ where $\mathbb{F}$ is a finite field of characteristic p: The resulting modular forms are called **mod p modular forms**.
- The following fact perhaps helps you get a feeling for mod p modular forms:

  Lemma 10.4. *For weight $k \geq 2$, any mod p modular form is the reduction of a modular form over a number field modulo p. In particular, in this case*
$$M_k(\Gamma_1(N); \mathbb{F}) = M_k(\Gamma_1(N); \mathbb{Z}[\tfrac{1}{N}]) \otimes_{\mathbb{Z}} \mathbb{F}.$$

  Proof. Using Riemann–Roch, one can show that $H^1(X^*_{\Gamma_1(N)}, \omega^{\otimes k}) = 0$. □

- However, it is often better to have a more intrinsic definition. For example:

Example of mod p modular form: The Hasse invariant

- Recall: For any elliptic curve in characteristic p, have Verschiebung isogeny
$$V : E^{(p)} \to E.$$
- In particular, have this for the universal elliptic curve.
- On relative differentials, this defines a pullback map in the other direction
$$V^* : \omega \to \omega^{\otimes p}.$$
- Tensoring with ω^{-1} yields
$$V^* : \mathcal{O} \to \omega^{\otimes(p-1)} \rightsquigarrow V^* \in \Gamma(\omega^{\otimes(p-1)}) = M_{p-1}(\Gamma_1(N); \mathbb{F}).$$
- This is the **Hasse invariant**, denoted by Ha.
- It is a mod p modular form of weight $p-1$. In fact: an eigenform of level 1.

Proposition 10.5. *The Hasse invariant has constant q-expansion $= 1 \in \mathbb{F}_p[[q]]$.*

Proof. Sketch: Use $\omega_{T(q)} = \omega_{\mathbb{G}_m}$: On $\mathbb{G}_m$ have $F = [p]$ and thus $V = 1$. □

- Note: $1 \in M_0(\Gamma_1(N); \mathbb{F}_p)$ is also a mod p modular form with q-expansion 1.
- This does not contradict the q-expansion principle as the weights are different.

Upshot: In characteristic p, we have a new phenomenon:

- Modular forms of different weight can have same q-expansion!
- E.g. for any modular form f of weight k, any $n \in \mathbb{N}$, have modular form $\mathrm{Ha}^n \cdot f$ of weight $k + n(p-1)$ with same q-expansion.
- In particular, this preserves eigenforms $\rightsquigarrow$ a system of eigenvalues for Hecke operators can correspond to eigenforms of different weights.
- The multiplication map

$$M_k(\Gamma_1(N); S) \xrightarrow{\cdot \mathrm{Ha}} M_{k+p-1}(\Gamma_1(N); S)$$

  is injective (q-expansion principle) but not surjective!
- Reason: Ha has zeros, precisely on supersingular points of $X^*_{\Gamma_1(N)}(\overline{\mathbb{F}}_p)$
- This is one motivation for removing those points, which yields the theory of p-adic modular forms.

Eisenstein series

- There is a more classical perspective on this that we want to at least mention:

  Definition 10.6.
  For any even $k \geq 4$, let

$$G_k := \sum_{\substack{(n,m)\in\mathbb{Z}^2 \\ (n,m)\neq(0,0)}} \frac{1}{(m+n\tau)^k}.$$

  This is a modular eigenform of weight k and level 1 with q-expansion

$$2\zeta(k)(1 - \sum_{n=0}^{\infty} \frac{4}{B_k}\sigma_{k-1}(n)q^n)$$

  where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and B_k is the k-th Bernoulli number.
  Define the normalised Eisenstein series as

$$E_k := \frac{1}{2\zeta(k)} G_k = 1 - \frac{4}{B_k}\sum_{n=0}^{\infty} \sigma_{k-1}(n)q^n \in \mathbb{Z}_{(p)}[[q]]$$

Lemma 10.7. *Assume $p \geq 5$, then in terms of q-expansions, we have*

$$E_{p-1} \equiv 1 \bmod p.$$

Proof. This follows from Kummer's congruences for Bernoulli numbers. □

Upshot: For $p \geq 5$, the reduction mod p of the Eisenstein series E_{p-1} is $=$ Ha:

$$E_{p-1} \equiv \mathrm{Ha} \bmod p \quad \text{“Deligne's congruence”}$$

Strategy (Deligne–Serre):

- Given eigenform of weight 1, reduce mod p.
- Multiply with Ha so it becomes eigenform of weight ≥ 2.
- Use construction from last lecture in this case.

Galois reps attached to mod p modular forms

Theorem 10.8. *Let $k \geq 1$ and let $f \in M_k(\Gamma_1(N), \varepsilon; \overline{\mathbb{F}}_p)$ be a mod p cuspidal eigenform. Let $\mathbb{F}_f$ be the (finite) field generated over $\mathbb{F}_p$ by the $a_p(f)$ and ε. Then there is a semi-simple Galois representation*

$$\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_f),$$

unramified outside $N\ell$ such that for all $\ell \nmid Np$,

$$\mathrm{charpoly}(\mathrm{Frob}_p | \rho_{\overline{f}}) = X^2 - a_p X + \varepsilon(p) p^{k-1}.$$

Step 1: Reduce to case of $k \geq 2$: If $k = 1$, multiply with Ha. This does not change the q-expansion, hence is still an eigenform. But its weight is ≥ 2.

Step 2: Lift the Hecke eigensystem to characteristic 0. This is always possible in weight ≥ 2, as we shall discuss next.

Deligne–Serre lifting lemma

- Let R be a $\mathbb{Z}[\frac{1}{N}]$-algebra without zero-divisors. Let
$$\mathbb{T}_R \subseteq \mathrm{End}(M_k(\Gamma_1(N); R))$$
be the Hecke algebra: R-subalgebra generated by the Hecke operators T_l, S_l.
- A Hecke eigensystem is a ring homomorphism
$$\psi : \mathbb{T}_R \to R.$$

The archetypal example of a Hecke eigensystem is:

- Let g be an eigenform in $M_k(\Gamma_1(N); R)$, this gives rise to a Hecke eigensystem
$$\psi_g : \mathbb{T}_R \to R, \quad T \mapsto a_T \quad \text{s.t. } T(g) = a_T g.$$

Lemma 10.9. *Assume that $R = K$ is a field. Then any Hecke eigensystem*

$$\psi : \mathbb{T}_K \to K$$

comes from a unique Hecke eigenform.

Proof.
- $\mathbb{T}_K$ is an Artinian K-algebra.
- After passing to extension, it is the product of Artinian local rings $\mathbb{T}_i$.
- Since $\mathbb{T}_K$ acts faithfully on $M := M_k(\Gamma_1(N); K)$, the submodule $M_i := \mathbb{T}_i \cdot M$ is non-zero (commutative algebra fact).
- Take any eigenvector in M_i. Then this has eigensystem ψ.
- Uniqueness: The Hecke eigenvalues determine the q-expansion. □

- Let K be a number field, $\mathfrak{p}$ a place over $p \nmid N$. Let $\mathbb{F}_{\mathfrak{p}}$ be the residue field.
- Write $\mathcal{O}_{K,(\mathfrak{p})}$ for the valuation subring of K of $\mathfrak{p}$-integral elements.

Lemma 10.10 (Deligne–Serre Lifting Lemma)**.** *Let $k \geq 2$ and let $g \in M_k(\Gamma_1(N); \mathbb{F}_{\mathfrak{p}})$ be an eigenform. Then there is a finite extension $K'|K$, an extension $\mathfrak{p}'|\mathfrak{p}$ and an eigenform $\tilde{g} \in M_k(\Gamma_1(N); \mathcal{O}_{K',(\mathfrak{p}')})$ such that $\tilde{g}$ reduces to g mod $\mathfrak{p}'$.*

Proof.
- Consider the ring homomorphism
$$\mathbb{T}_{\mathcal{O}_K} \to \mathbb{T}_{\mathbb{F}_{\mathfrak{p}}} \xrightarrow{\psi_g} \overline{\mathbb{F}}_p.$$
- This corresponds to a maximal ideal $\mathfrak{m}$. Since
$$\mathrm{Spec}(\mathbb{T}_{\mathcal{O}_K}) \to \mathrm{Spec}(\mathcal{O}_K)$$
is locally free, can find prime ideal $\mathfrak{q} \subseteq \mathfrak{m}$ such that $\mathfrak{q} \cap \mathcal{O} = 0$.

- This defines a non-zero prime ideal of $\mathbb{T}_K$, corresponding to Hecke eigensystem
$$\mathbb{T}_K \to K'.$$
- By Lemma Lemma 10.9, this corresponds to an eigenform.
- Use uniqueness in Lemma Lemma 10.9 to see that this reduces to g. □

Proof of Theorem Theorem 10.8

- Let f be mod p eigenform of weight ≥ 2, defined over extension $\mathbb{F}_f$ of $\mathbb{F}_p$.
- Lift to eigenform $\tilde{f}$ over some number field K. Let $\mathfrak{p}$ be a place over p.
- Associated to this, have Galois representation
$$\rho_{\tilde{f}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_{\mathfrak{p}}).$$
- Since image is compact, can always find isomorphic representation of the form
$$\rho_{\tilde{f}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}_{K,\mathfrak{p}}).$$
- Reduce this mod $\mathfrak{p}$ to get representation
$$\overline{\rho}_{\tilde{f}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_{\mathfrak{p}}).$$
- This might not yet be semi-simple $\rightsquigarrow$ Define ρ_f as semi-simplification
$$\rho_f := \overline{\rho}_{\tilde{f}}^{ss}$$
(:=direct sum of Jordan–Hölder factors). Same trace and determinant as $\overline{\rho}_{\tilde{f}}$.
- Since $\ker \rho_f \subseteq \ker \rho_{\tilde{f}}$, this is unramified outside of pN.
- Remains to show: ρ_f already defined over $\mathbb{F}_f$: For this, STP:
$$\rho_f = \sigma \circ \rho_f \quad \text{for all } \sigma \in \mathrm{Gal}(\mathbb{F}_{\mathfrak{p}}|\mathbb{F}_f).$$
- Certainly, all characteristic polynomials of Frob_ℓ for $\ell \nmid pN$,
$$X^2 - a_\ell X + \varepsilon(\ell),$$
are preserved by σ, since $a_\ell, \varepsilon(\ell) \in \mathbb{F}_f$ by definition. Now use:

  Theorem 10.11 (Brauer–Nesbitt)**.** *Let F be a perfect field. Let G be a finite group. Let V, W be two semi-simple finite dimensional representations of G over F. Then $V \cong W$ if and only if*
$$\mathrm{charpoly}(g|V) = \mathrm{charpoly}(g|W) \quad \text{for all } g \in G.$$
- This replaces the similar statement on traces from Section 6 if the coefficients are of positive characteristic.
- This finishes the construction of $f \rightsquigarrow \rho_f$. □

Serre's conjecture

- A quick digression before we continue with weight 1 forms:
- In 73', Serre gave a conjecture that described exactly which mod p Galois representations arise in this way. It became known as

Conjecture 10.12 (Serre's Modularity Conjecture, or Serre's Conjecture)**.** *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous Galois representation that is odd and irreducible. Then there is a mod p eigenform g such that*
$$\rho = \rho_g.$$

In 1987, Serre published an article refining this conjecture:

- He predicted the optimal level of g (in terms of the Artin conductor of ρ)
- He predicted the optimal weight (recipe in terms of ramification of ρ at p)
- He showed that his conjecture would imply Fermat's Last Theorem.
- Serre's Conjecture is now a theorem by Khare–Wintenberger (2008).

Lifting ρ_f to $\mathbb{C}$

- Back to our earlier setup: $f \in M_1(\Gamma_0(N), \varepsilon)$ eigenform
- K number field generated by eigenvalues and ε.
- Let $L :=$ set of primes of $\mathbb{Q}$ which decompose completely in K. This is an infinite set by Chebotarev.
- Let $\ell \in L$ and λ a place over ℓ in K. Let $\mathcal{O}_{K,(\lambda)} \subseteq K$ be λ-integral elements.
- Now apply previous section with role of p played by varying $\ell \in L$.
- For any such λ, consider reduction $\overline{f}$ of f mod λ. This is a mod ℓ eigenform.
- Theorem Theorem 10.8 associates a semi-simple Galois representation

$$\rho_{\overline{f},\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_\ell)$$

  (over $\mathbb{F}_\ell$ as λ completely split) with charpoly of Frob_p for any $p \nmid \ell N$ given by

$$\mathrm{charpoly}(\mathrm{Frob}_p | \rho_{\overline{f},\ell}) = X^2 - a_p X + \varepsilon(p).$$

- Zeros of this are n-th units roots, which we may assume are all in $\mathcal{O}_K$ (by the next lemma) and thus in $\mathbb{F}_l$. Thus

$$= (X - \overline{a})(X - \overline{b}) \quad \text{for some } \overline{a}, \overline{b} \in \mathbb{F}_\ell^\times.$$

Definition 10.13. *Let $G_\ell := \mathrm{im}(\rho_{\overline{f},\ell}) \subseteq \mathrm{GL}_2(\mathbb{F}_\ell)$. This is a finite group.*

Lemma 10.14. *If f is of weight 1, there is $A \in \mathbb{N}$ such that $|G_\ell| \leq A$ for all $\ell \in L$.*

Proof. After using a crucial estimate on the coefficients of weight 1 modular forms this becomes purely group theoretical: use classification of subgroups of $\mathrm{GL}_2(\mathbb{F}_\ell)$. We shall not discuss it, but if you want to see it: [DS74, Lemma 8.4]. □

- Enlarge K so that it contains all n-th unit roots for $n \leq A$. This shrinks L.
- Also exclude the primes $\leq A$ from L. Then L is still infinite.

Lemma 10.15. *Let G be a finite group of order prime to ℓ. Then any representation $\rho : G \to \mathrm{GL}_2(\mathbb{F}_\ell)$ lifts to a representation*

$$\tilde{\rho} : G \to \mathrm{GL}_2(\mathcal{O}_{K,(\lambda)}).$$

Proof. **Step 1: Lift to $\mathrm{GL}_2(\mathbb{Z}_\ell)$:**

- STP: any $G \to \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})$ lifts to $G \to \mathrm{GL}_2(\mathbb{Z}/\ell^{n+1}\mathbb{Z})$.
- For this, consider the short exact sequence

$$1 \to 1 + \ell^n M_2(\mathbb{F}_\ell) \to \mathrm{GL}_2(\mathbb{Z}/\ell^{n+1}\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \to 1$$

- The first group is isomorphic to $M_2(\mathbb{F}_\ell)$ via $1 + \ell^n x \mapsto x$.
- Now use non-abelian group cohomology: Consider the left-action

$$G \text{ acts on } \mathrm{Map}(G, \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \text{ via } \quad g \cdot \varphi := \varphi(g)\varphi(g)^{-1}.$$

- Then

$$\mathrm{Hom}(G, \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})) = \mathrm{Map}(G, \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}))^G.$$

- Apply Map to the above short exact sequence, then G gives a long exact sequence of non-abelian group cohomology

$$\cdots \to \mathrm{Hom}(G, \mathrm{GL}_2(\mathbb{Z}/\ell^{n+1})) \to \mathrm{Hom}(G, \mathrm{GL}_2(\mathbb{Z}/\ell^n)) \to H^1(G, \mathrm{Map}(G, M_2(\mathbb{F}_\ell)))$$

- Fact in group cohomology: $H^1(G, A) = 0$ if $|G|$, $|A|$ finite and coprime.

Step 2: Already defined over number field

- Since G is finite, ρ is semi-simple.
- Traces must be sums of roots of unity of order $\leq |G| \leq A \rightsquigarrow$ contained in K.
- Brauer–Nesbitt ensures that ρ is already defined over K.
- Finally, $K \cap \mathbb{Z}_\ell = \mathcal{O}_{K,(\lambda)}$ □

Back to prove of Theorem Theorem 10.2 (Galois representation for weight 1 forms)

- Apply this to $\rho_g : G_\ell \hookrightarrow \mathrm{GL}_2(\mathbb{F}_\ell)$, get $G_\ell \to \mathrm{GL}_2(\mathcal{O}_{K,(\lambda)})$
- Compose with $G_\mathbb{Q} \to G_\ell$ to get Galois representation

$$\tilde{\rho}_{g,\ell} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_{K,(\lambda)}).$$

Proposition 10.16. *The lift $\tilde{\rho}_{g,\ell} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_{K,(\lambda)})$ is such that for all $p \nmid \ell N$,*

$$\mathrm{charpoly}(\mathrm{Frob}_p|\tilde{\rho}_{g,\ell}) = X^2 - a_p X + \varepsilon(p).$$

The key is to vary ℓ: Recall that we get $\tilde{\rho}_{g,\ell}$ for all $\ell \in L$.

Definition 10.17. *Let Y be the set of polynomials of the form $(X-\alpha)(X-\beta)$ for $\alpha, \beta \in \mathcal{O}_{K,(\lambda)}$ roots of unity of order $\leq |A|$. This is a finite set.*

- Note: Given $n \leq A$, $\ell \in L$, and n-th root of unity $x \in \mathbb{F}_\ell^\times$, there is exactly one lift to root of unity in K (existence: $n \leq A$, uniqueness: $\ell \geq A$).
- Consequence: If $F, G \in Y$, then $F \equiv G \bmod \lambda$ implies $F = G$.
- Idea: Apply this to $F = X^2 - a_p X + \varepsilon(p)$ and $G = \mathrm{charpoly}(\mathrm{Frob}_p|\tilde{\rho}_{g,\ell})$

Lemma 10.18. $X^2 - a_p X + \varepsilon(p) \in Y$.

Proof. For all λ over a prime in L, there is $F \in Y$ for which

$$\mathrm{charpoly}(\mathrm{Frob}_p|\rho_{g,\ell}) \equiv F \bmod \lambda$$

since $\rho_{g,\ell}$ has finite image. Thus also

$$X^2 - a_p X + \varepsilon(p) \equiv F \bmod \lambda.$$

Since Y is finite, there is F for which this holds for infinitely many λ. Thus

$$X^2 - a_p X + \varepsilon(p) = F \in Y$$ □

Proof of Proposition.

- As G_ℓ finite of order $\leq A$, have

$$\mathrm{charpoly}(\mathrm{Frob}_p|\tilde{\rho}_{g,\ell}) \in Y$$

- $\mathrm{charpoly}(\mathrm{Frob}_p|\tilde{\rho}_{g,\ell}) \equiv \mathrm{charpoly}(\mathrm{Frob}_p|\rho_{g,\ell}) \equiv X^2 - a_p X + \varepsilon(p) \bmod \lambda$
- Thus both sides are in Y, which implies

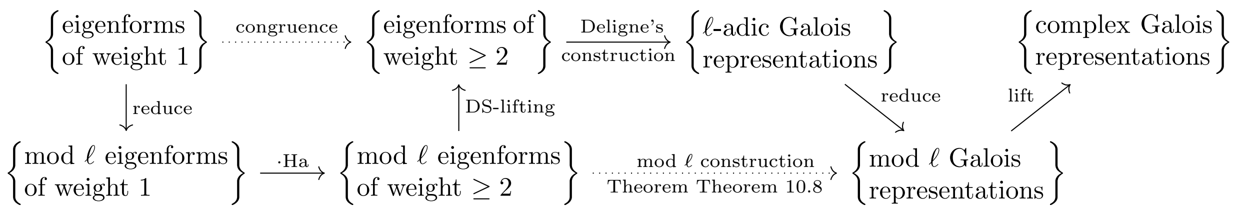
$$\mathrm{charpoly}(\mathrm{Frob}_p|\rho_{g,\ell}) = X^2 - a_p X + \varepsilon(p).$$ □

Remains to prove:

- ρ is odd: This is simply because ε is odd and c has order 2.
- ρ is irreducible: Use complex analytic estimate due to Rankin □

This finishes the construction of eigenforms $\rightsquigarrow$ Galois representations!

Summary of the weight 1 construction



11. The Langlands program for general groups, part I

Next aims:

- More or less precise statements (for parts of) Langlands program for GL_n, or even general G reductive over $\mathbb{Q}$.

Langlands(-Clozel-Fontaine-Mazur) reciprocity for $\mathrm{GL}_{n,F}$:

- Let F be an arbitrary number field.
- Fix some prime ℓ and an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$.
- The Langland(-Clozel-Fontaine-Mazur) reciprocity conjecture states that for any $n \geq 1$, there exists a (unique) bijection between
  - i) the set of L-algebraic cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$,
  - ii) the set of (isomorphism classes) of irreducible continuous representations $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which are almost everywhere unramified, and de Rham at places dividing ℓ,

  such that the bijection matches Satake parameters with eigenvalues of Frobenius elements.

Comments:

- This is only a part of the Langlands program for $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{n,F}$.
- This does not describe all automorphic representations for $\mathrm{GL}_{n,F}$, nor all Galois representations.
- Not known for $F = \mathbb{Q}, n \geq 2$, e.g., there exists L-algebraic cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ (associated to Maassforms), which should correspond to *even* irreducible, 2-dimensional ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (with finite image).
- Langlands-Tunnell: If the image is solvable, automorphicity of the Galois representation is known, cf. [GH19, Theorem 13.4.5.]. This does not cover almost everywhere unramified 2-dimensional, irreducible Galois representations with image $\mathrm{SL}_2(\mathbb{F}_5)$.
- Unicity follows from the (strong) multiplicity one theorems on both sides.
- Today: Explain what "L-algebraic" means.

Automorphic forms for general groups:

- G an arbitrary reductive group over $\mathbb{Q}$.
- Will introduce a convenient (=more algebraic) replacement for $L^2([G])$.
- Recall for $G = \mathrm{GL}_2 = \mathrm{GL}_{2,\mathbb{Q}}$, $k \in \mathbb{Z}$, the embedding
$$\Phi\colon M_k \to C^\infty(\mathrm{GL}_2(\mathbb{A})),\ f \mapsto \varphi_f$$
  from Section 3.
- Recall the description of the image of $\Phi(M_k)$ resp. $\Phi(S_k)$, namely:
  - For $k \in \mathbb{Z}$ an element $\varphi \in C^\infty(\mathrm{GL}_2(\mathbb{A}))$ lies in the image of
$$\Phi\colon M_k \to C^\infty(\mathrm{GL}_2(\mathbb{A}))$$
    if and only if
    - $\varphi(g, g_\infty z) = z^{-k}\varphi(g, g_\infty)$ for $(g, g_\infty) \in \mathrm{GL}_2(\mathbb{A})$, $z \in \mathbb{C}^\times \subseteq \mathrm{GL}_2(\mathbb{R})$.
    - φ is of moderate growth (implies the vanishing of negative Fourier coefficients).
    - For each $g \in \mathrm{GL}_2(\mathbb{A}_f)$
$$Y * \varphi(g, -) = 0,$$

where $Y \in \mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_2(\mathbb{R})_{\mathbb{C}}$ is a suitably constructed, natural element (implies holomorphicity).

  * $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \mathrm{GL}_2(\mathbb{Q})$.

- f is moreover cuspidal if and only if

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi_f(ng)dn = 0$$

for all unipotent radicals N in proper parabolics $P \subseteq G$ (defined over $\mathbb{Q}$).

- Fix a maximal compact (usually not connected) subgroup K_∞ of $G(\mathbb{R})$, e.g., $\mathrm{O}_2(\mathbb{R}) = \mathrm{SO}_2(\mathbb{R}) \cup s\mathrm{SO}_2(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{R})$ with $s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Definition 11.1 (cf. [GH19, Definition 6.5]). *Let G be a reductive group over $\mathbb{Q}$. An (adélic) automorphic form for G is a smooth function $\varphi\colon G(\mathbb{A}) \to \mathbb{C}$ such that*

1) *φ is of moderate growth (was defined in Section 4, not important for this lecture).*
2) *φ is right K_∞-finite, i.e., the functions $g \mapsto \varphi(gk)$ for $k \in K_\infty \subseteq G(\mathbb{A})$ span a finite dimensional vector space.*
3) *φ is killed by an ideal in $Z(\mathfrak{g}_{\mathbb{C}})$ of finite codimension, where $\mathfrak{g}_{\mathbb{C}}$ denotes the (complexified) Lie algebra of $G(\mathbb{R})$, and $Z(\mathfrak{g}_{\mathbb{C}})$ the center of the enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$.*
4) *φ is left $G(\mathbb{Q})$-invariant, i.e., $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in G(\mathbb{Q})$.*

The Casimir element for GL_2:

- Let us check that 3) is indeed verified for $\varphi_f\colon C^\infty(\mathrm{GL}_2(\mathbb{A})) \to \mathbb{C}$ with $f \in M_k$.
- Namely, the element Y was not in $Z(\mathfrak{gl}_{2,\mathbb{C}})$.
- Set

$$H := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X := \frac{1}{2}\begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix}, \quad Y := \frac{1}{2}\begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}, \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Then
  1)
  $$H * \varphi_f = -k\varphi_f$$
  as $z \cdot \varphi_f = z^{-k}\varphi_f$ for $z \in \mathbb{C}^\times$.
  2) $Y * \varphi_f = 0$.
- $[X, Y] = XY - YX = H$ (because H, X, Y form an $\mathfrak{sl}_2$-triple)
- Define the Casimir operator

$$\Delta := \frac{1}{4}(H^2 + 2XY + 2YX).$$

Then $\Delta \in Z(\mathfrak{gl}_{2,\mathbb{C}})$.

- In fact, $Z(\mathfrak{gl}_{2,\mathbb{C}})$ is generated, as a $\mathbb{C}$-algebra, by Δ and $Z \in \mathfrak{gl}_{2,\mathbb{C}}$ (this follows from the Harish-Chandra isomorphism, cf. [GH19, Theorem 4.6.1]).
- One calculates (using $Y * \varphi_f = 0$)

$$\Delta * \varphi_f = \frac{1}{4}(H^2 - 2XY + 2YX) * \varphi_f = \frac{1}{4}(H^2 - 2H) * \varphi_f = \frac{1}{4}(k^2 - 2k)\varphi_f,$$

thus φ_f is indeed killed by the ideal $\langle \Delta - \frac{1}{4}(k^2 - 2k), Z + k\rangle \subseteq Z(\mathfrak{gl}_{2,\mathbb{C}})$ of finite codimension.

More on automorphic forms:

- G arbitrary reductive over $\mathbb{Q}$.
- Set $\mathcal{A}(G)$ as the space of automorphic forms, and $\mathcal{A}([G])$ as the subspace of automorphic forms, which are invariant under A_G (acting via left or right translations on $G(\mathbb{A})$).
- Unfortunately, the spaces $\mathcal{A}([G])$ and $\mathcal{A}(G)$ are not stable under $G(\mathbb{R})$, because K_∞-finiteness need not be preserved.

Example that K_∞-finiteness is not preserved (following Andreas Mihatsch):

- Consider $G = \mathrm{GL}_2(\mathbb{R})$, $K := K_\infty^\circ = \mathrm{SO}_2(\mathbb{R}) \cong S^1$, and V any representation on functions on G containing $C_c^\infty(G)$. We give an example showing that K-finiteness is not preserved under the action by right translation.
- More precisely, there exists $\varphi \in C_c^\infty(G)$, and $h \in G$ such that φ is K-finite, but not $K_h := h^{-1}Kh$-finite.
- For this: Pick $h \in G$, such that $K_h \cap K = \{\pm 1\}$.
- Then choose a non-zero $\varphi \in C_c^\infty(G)$ invariant under K, such that $\varphi \equiv 0$ on an open subset of K_h (e.g., construct φ by pullback a function from G/K, note that the image of K_h in G/K is of dimension > 0).
- No non-zero K_h-finite function on K_h can vanish on an open subset. Indeed, the K_h-finite functions on $K_h \cong S^1$ are precisely the restrictions of Laurent-polynomials.

More on automorphic forms (continued):

- Clearly, $G(\mathbb{A}_f)$ acts on $\mathcal{A}(G)$ via $\varphi \mapsto (g \mapsto \varphi(gh))$ for $h \in G(\mathbb{A}_f)$.
- Similarly, K_∞, even $A_G K_\infty$, acts on $\mathcal{A}(G)$
- Less clear, but true (cf. [GH19, Proposition 4.4.2.]): $\mathfrak{g}_\mathbb{C}$ acts on $\mathcal{A}(G)$.

$\Rightarrow$ $U(\mathfrak{g}_\mathbb{C})$, in particular $Z(\mathfrak{g}_\mathbb{C}) \subseteq U(\mathfrak{g}_\mathbb{C})$, acts on $\mathcal{A}(G)$ by differential operators.

- The $\mathfrak{g}_\mathbb{C}$ and K_∞-action make $\mathcal{A}(G)$ into a $(\mathfrak{g}_\mathbb{C}, K_\infty)$-module, in the following sense.

Definition 11.2 (cf. [GH19, Definition 4.5.]). *A $(\mathfrak{g}_\mathbb{C}, K_\infty)$-module is a $\mathbb{C}$-vector space V, which is simultaneously a $\mathfrak{g}_\mathbb{C}$- and K_∞-module, such that*

1) *V is a union of finite dimensional K_∞-stable subspaces.*
2) *For $X \in \mathfrak{k} := \mathrm{Lie}(K_\infty)_\mathbb{C}$ and $\varphi \in V$ we have*

$$X * \varphi = \frac{\partial}{\partial t}(\exp(tX)\varphi)_{t=0}.$$

3) *For $h \in K_\infty$, $X \in \mathfrak{g}_\mathbb{C}$ and $\varphi \in V$ we have*

$$h(X * (h^{-1}\varphi) = (\mathrm{Ad}(h)(X)) * \varphi.$$

Comments:

- We required no topology on V, thus $(\mathfrak{g}_\mathbb{C}, K_\infty)$-modules are much more algebraic than $G(\mathbb{R})$-representations, and thus easier to handle.
- In 2) the limit is taken for the natural topology of a finite dimensional K_∞-stable subspace of V containing φ.
- For each representation of $G(\mathbb{R})$ on some Hilbert space $\mathbb{C}$-vector space, the space of smooth and K_∞-finite vectors is naturally a $(\mathfrak{g}_\mathbb{C}, K_\infty)$-module, cf. [GH19, Proposition 4.4.2.], and on *unitary* representations one does not loose any informations, cf. [GH19, Theorem 4.4.4.], namely: isomorphism

classes of irreducible *unitary* representations of $G(\mathbb{R})$ inject into isomorphism classes of irreducible $(\mathfrak{g}_{\mathbb{C}}, K_\infty)$-modules.

- Moreover, the $(\mathfrak{g}_{\mathbb{C}}, K_\infty)$-modules obtained in this way are *admissible*, i.e., for each (finite dimensional) irreducible representation σ of K_∞, the σ-isotypic component is finite dimensional, cf. [GH19, Theorem 4.4.1.].
- Version of Schur's lemma $\Rightarrow$ On each irreducible, admissible $(\mathfrak{g}_{\mathbb{C}}, K_\infty)$-module π the center $Z(\mathfrak{g}_{\mathbb{C}})$ of $U(\mathfrak{g}_{\mathbb{C}})$ acts via some morphism
$$\omega_\pi \colon Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$$
of $\mathbb{C}$-algebras, called the "infinitesimal character of π".
- The Harish-Chandra isomorphism, cf. [GH19, Theorem 4.6.1.], furnishes
$$Z(\mathfrak{g}_{\mathbb{C}}) \cong U(\mathfrak{t}_{\mathbb{C}})^W$$
for each Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ and W the corresponding Weyl group. Note $U(\mathfrak{t}_{\mathbb{C}}) \cong \mathrm{Sym}^\bullet \mathfrak{t}_{\mathbb{C}}$ because $\mathfrak{t}_{\mathbb{C}}$ is an abelian Lie algebra.
- As W is finite, $\mathbb{C}$-algebra morphisms
$$Z(\mathfrak{g}_{\mathbb{C}}) \cong U(\mathfrak{t}_{\mathbb{C}})^W \to \mathbb{C}$$
correpond bijectively to W-orbits of $\mathbb{C}$-algebra homomorphisms
$$U(\mathfrak{t}_{\mathbb{C}}) \cong \mathrm{Sym}^\bullet \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}.$$
Moreover,
$$\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(U(\mathfrak{t}_{\mathbb{C}}), \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}-\mathrm{lin}}(\mathfrak{t}_{\mathbb{C}}, \mathbb{C}) =: \mathfrak{t}_{\mathbb{C}}^\vee.$$
$U(\mathfrak{t}_{\mathbb{C}})^W \to U(\mathfrak{t}_{\mathbb{C}})$ is finite.
- Thus, the infinitesimal character defines a canonical W-orbit of elements in the "weight space" $\mathfrak{t}_{\mathbb{C}}^\vee$.
- Now assume that $\mathfrak{t} = \mathrm{Lie}(T)$ for some maximal torus $T \subseteq G_{\mathbb{R}}$. Then W-equivariantly
$$X^*(T_{\mathbb{C}}) \subseteq \mathfrak{t}_{\mathbb{C}}^\vee.$$
- An irreducible, admissible $(\mathfrak{g}_{\mathbb{C}}, K_\infty)$-module π is called L-algebraic if the W-orbit in $\mathfrak{t}_{\mathbb{C}}^\vee$ corresponding to the infinitesimal character ω_π lies in the finite free $\mathbb{Z}$-module $X^*(T_{\mathbb{C}}) \subseteq \mathfrak{t}_{\mathbb{C}}^\vee$, cf. [BG10, Definition 2.3.1.], [Tho, Definition 93].

$(\mathfrak{gl}_{2,\mathbb{C}}, \mathrm{O}_2(\mathbb{R}))$-modules generated by modular forms:

- Let $f \in M_k$. We will analyze the $(\mathfrak{gl}_{2,\mathbb{C}}, \mathrm{O}_2(\mathbb{R}))$-module $V \subseteq \mathcal{A}(\mathrm{GL}_2)$ generated by $\varphi := \varphi_f$.
- Recall
$$Y * \varphi = 0, \quad z \cdot \varphi = z^{-k}\varphi \text{ for } z \in \mathbb{C}^\times.$$
- It is not difficult to see that there exists a unique, up to isomorphism, $(\mathfrak{gl}_{2,\mathbb{C}}, \mathrm{O}_2(\mathbb{R}))$-module D'_{k-1} (the dual of D_{k-1} in [Del73, Section 2.1]) generated by such an element φ, and that $D'_{k-1} \cong V$ is irreducible.
- In particular, it only depends on k, not f.
- We obtain, as was mentioned some time ago,
$$M_k \cong \mathrm{Hom}_{(\mathfrak{gl}_{2,\mathbb{C}}, \mathrm{O}_2(\mathbb{R}))}(D'_{k-1}, \mathcal{A}(\mathrm{GL}_2)),$$
(note: the space $\mathcal{H}(G_{\mathbb{A}})$ in [Del73, Scholie 2.1.3.] agrees with our $\mathcal{A}(\mathrm{GL}_2)$ only up to inversion on $\mathrm{GL}_2(\mathbb{A})$).

- We calulated
$$\Delta * \varphi = \frac{1}{4}(k^2 - 2k)\varphi,\ Z * \varphi = -k\varphi,$$
and this determines the infinitesimal character $\omega_{D'_{k-1}}$.
- The W-orbit of $\mathbb{C}$-algebra homomorphisms $U(\mathfrak{t}_{\mathbb{C}}) \cong \mathbb{C}[Z, H] \to \mathbb{C}$ corresponding to $\omega_{D'_{k-1}}$ under the Harish-Chandra isomorphism is given by the homomorphisms
$$Z \mapsto -k,\ H \mapsto \pm(k-1).$$
- The lattice $X^*(T) \subseteq \langle Z, H\rangle_{\mathbb{C}}^{\vee}$ is given by $\mathbb{C}$-linear maps
$$Z \mapsto a,\ H \mapsto b$$
with $a, b \in \mathbb{Z}$ *and* $a \equiv b \bmod 2$.
- Thus, D'_{k-1} is *not* L-algebraic (it is C-algebraic in the sense of [BG10]).
- However,
$$D'_{k-1} \otimes_{\mathbb{C}} |\det|^{1/2+a}$$
is L-algebraic for any $a \in \mathbb{Z}$ because (the $(\mathfrak{gl}_{2,\mathbb{C}}, \mathrm{O}_2(\mathbb{R}))$-module associated with) $|\det|^{1/2}$ has infinitesimal character corresponding to
$$Z \mapsto 1,\ H \mapsto 0.$$

Upshot:

- For any G there exists the space $\mathcal{A}(G)$ of (adélic) automorphic forms, which for GL_2 naturally contains the φ_f with $f \in M_k$.
- The space $\mathcal{A}(G)$ carries a $(\mathfrak{g}_{\mathbb{C}}, K_\infty) \times G(\mathbb{A}_f)$-action, and is a more algebraic replacement for $L^2([G])$ with its $G(\mathbb{A})$-action.

A new notion of automorphic representations:

- We redefine the notion of an automorphic representation using $\mathcal{A}(G)$ instead of $L^2([G])$.
- Namely, an automorphic representation for G is an irreducible $G(\mathbb{A}_f) \times (\mathfrak{g}_{\mathbb{C}}, K_\infty)$-subquotient of $\mathcal{A}(G)$, cf. [GH19, Definition 6.8.].
- The former automorphic representations will from now on be called "L^2-automorphic representations".
- For informations how both notions relate, cf. [GH19, Section 6.5.].

Flath's theorem (cf. [GH19, Theorem 5.7.1.]**):**

- For almost all primes p the reductive group $G_{\mathbb{Q}_p} := G \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{Q}_p)$ is "unramified", i.e., extends to a reductive group scheme
$$\mathcal{G}_p \to \mathrm{Spec}(\mathbb{Z}_p).$$
- Equivalently, $G_{\mathbb{Q}_p}$ is quasi-split (=contains a Borel subgroup defined over $\mathbb{Q}_p$) and split(=contains a maximal and split torus) over an unramified extension.
- Then
$$G(\mathbb{A}) = \prod_p{}' (G(\mathbb{Q}_p), \mathcal{G}_p(\mathbb{Z}_p)) \times G(\mathbb{R}),$$
where $\mathcal{G}_p(\mathbb{Z}_p) \subseteq G(\mathbb{Q}_p)$ is a compact-open subgroup (cf. [GH19, Proposition 2.3.1.]), called "hyperspecial".

- Flath's theorem (cf. [GH19, Section 5.7.] and [Fla79]) states that irreducibe, admissible $G(\mathbb{A}_f)\times(\mathfrak{g}_{\mathbb{C}},K_\infty)$-modules π decompose (uniquely) into a "restricted tensor product"

$$\pi \cong \bigotimes_{p \text{ prime}}{}' \pi_p \otimes \pi_\infty$$

of irreducible, smooth (even admissible) $G_{\mathbb{Q}_p}$-representations π_p and an irreducible, admissible $(\mathfrak{g}_{\mathbb{C}},K_\infty)$-module π_∞.
- Moreover, for almost all primes p the representation π_p is unramified, i.e., the group $G_{\mathbb{Q}_p}$ is unramified and $\pi_p^{\mathcal{G}_p(\mathbb{Z}_p)} \neq 0$, where $\mathcal{G}_p$ is a reductive model of $G_{\mathbb{Q}_p}$.
- Important, cf. [GH19, Theorem 7.5.1.]: $\mathcal{H}(G(\mathbb{Q}_p),\mathcal{G}_p(\mathbb{Z}_p))$ is again commutative! ($\Rightarrow$ If $\pi_p^{\mathcal{G}_p(\mathbb{Z}_p)} \neq 0$, then its dimension is one.)

Upshot:

- Thus irreducible, admissible $G(\mathbb{A}_f)\times(\mathfrak{g}_{\mathbb{C}},K_\infty)$-modules are a collection of "local data".
- Being automorphic puts strong relations among these local data.
- For almost all primes p get homomorphism

$$\chi_p\colon \mathcal{H}(G(\mathbb{Q}_p),\mathcal{G}_p(\mathbb{Z}_p)) \to \mathbb{C},$$

similarly to case for GL_2 $\Rightarrow$ this will yield analog of a "system of Hecke eigenvalues".
- For π set

$$\pi_f := \bigotimes_p{}' \pi_p,$$

thus

$$\pi \cong \pi_f \otimes \pi_\infty.$$

Definition 11.3 (L-algebraic automorphic representation)**.** *An automorphic representation*

$$\pi \cong \pi_f \otimes \pi_\infty$$

is L-algebraic if the irreducible, admissible $(\mathfrak{g}_{\mathbb{C}},K_\infty)$*-module* π_∞ *is L-algebraic, cf.* [BG10, Definition 3.1.1.].

Examples:

- An automorphic representation $\pi \subseteq \mathcal{A}(\mathrm{GL}_2)$ which is generated by a modular form is not L-algebraic, but the twist

$$\pi \otimes_{\mathbb{C}} |\det|_{\text{adélic}}^{1/2+a}$$

is for any $a \in \mathbb{Z}$.
- A continuous character $\psi\colon \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}$ is L-algebraic if and only if

$$\psi = \chi|-|_{\text{adélic}}^k$$

for some $k \in \mathbb{Z}$, $|-|_{\text{adélic}}$ the adélic norm and $\chi\colon \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}$ a character of finite order.

12. The Langlands program for general groups, part II

Last time:

- F arbitrary number field
- The Langlands(-Clozel-Fontaine-Mazur) reciprocity conjecture for $\mathrm{GL}_{n,F}$ says:
  - ℓ some prime. Fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$.
  - Then for any $n \geq 1$, there exists a (unique) bijection between
    - i) the set of L-algebraic cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$,
    - ii) the set of (isomorphism classes) of irreducible continuous representations $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which are almost everywhere unramified, and de Rham at places dividing ℓ,

    such that the bijection matches Satake parameters with eigenvalues of Frobenius elements.
- Introduced space of (adélic) automorphic forms $\mathcal{A}(G)$ for any G.
- Flath's theorem $\Rightarrow$ automorphic representations π factorize: $\pi \cong \bigotimes_p' \pi_p \otimes \pi_\infty$.
- Explained "L-algebraic" (=infinitesimal character of π_∞ is "integral").

Today:

- Introduce "cuspidal" automorphic representations.
- Explain Satake parameters.
- Discuss some expectations of the Langlands program for general G.

Definition 12.1 ([GH19, Definition 9.2])**.** *Let $G/\mathbb{Q}$ be reductive. We call $\varphi \in L^2([G])$ resp. $\varphi \in \mathcal{A}(G)$ cuspidal if*

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)dn = 0$$

for all unipotent radicals N of proper parabolic subgroups $P \subseteq G$ (defined over $\mathbb{Q}$), and almost all $g \in G(\mathbb{A})$.

Cuspidal automorphic representations:

- Set $L^2_{\mathrm{cusp}}([G])$ resp. $\mathcal{A}_{\mathrm{cusp}}(G)$ resp. $\mathcal{A}_{\mathrm{cusp}}([G])$ as the subspaces of cuspidal elements.
- There is an embedding with dense image

  $$\mathcal{A}_{\mathrm{cusp}}([G]) \subseteq L^2_{\mathrm{cusp}}([G]),$$

  i.e., cuspidal automorphic forms satisfy a growth condition strong enough to make them L^2 (they are "rapidly decreasing"), cf. [GH19, Section 6.5.].
- An L^2-automorphic representation of $G(\mathbb{A})$ is cuspidal if it is isomorphic to a subquotient (actually subrepresentation) of $L^2_{\mathrm{cusp}}([G])$.
- Similarly, an automorphic representation is cuspidal if it occurs as a subquotient of $\mathcal{A}_{\mathrm{cusp}}(G)$.
- Gelfand, Piatetski-Shapiro: As a unitary $G(\mathbb{A})$-representation

  $$L^2_{\mathrm{cusp}}([G]) \cong \widehat{\bigoplus_{\pi \in \widehat{G(\mathbb{A})}}} m_\pi \pi$$

  with each m_π finite, i.e., $m_\pi \in \mathbb{N} \cup \{0\}$, cf. [GH19, Corollary 9.1.2].

- In particular, $L^2_{\mathrm{cusp}}([G]) \subset L^2_{\mathrm{disc}}([G])$ (note: the trivial representation occurs in $L^2_{\mathrm{disc}}([G]) \setminus L^2_{\mathrm{cusp}}([G])$).
- From

$$L^2_{\mathrm{cusp}}([G]) \cong \widehat{\bigoplus_{\pi \in \widehat{G(\mathbb{A})}}} m_\pi \pi$$

on deduces the decomposition (into irreducible, admissible $G(\mathbb{A}_f) \times (\mathfrak{g}_{\mathbb{C}}, K_\infty)$-modules)

$$\mathcal{A}_{\mathrm{cusp}}([G]) \cong \bigoplus_{\pi \in \widehat{G(\mathbb{A})}} m_\pi \pi^{K_\infty-\mathrm{finite}},$$

where $\pi^{K_\infty-\mathrm{finite}} \subseteq \pi$ denotes the (dense) subspace of K_∞-finite (in particular, smooth) vectors, cf. [GH19, Theorem 4.4.4.].
- Thus, the decomposition of $L^2_{\mathrm{cusp}}([G])$ is "the same as" the decomposition of $\mathcal{A}_{\mathrm{cusp}}([G])$.
- Piatetski-Shapiro, Shalika, cf. [GH19, Theorem 11.3.4]: F number field, $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{n,F}$. Then the $G(\mathbb{A})$-representation on $L^2_{\mathrm{cusp}}([G])$ is multiplicity free ("multiplicity one").
- "Strong multiplicity one": Assume π, π' are cuspidal automorphic representations for $\mathrm{GL}_{n,F}$. If $\pi_p \cong \pi'_p$ for almost all primes p, then $\pi \cong \pi'$, cf. [GH19, 11.7.2].
- Not true without cuspidality, cf. [BG10, page 38].
- Moeglin, Waldspurger, cf. [GH19, Theorem 10.7.1.], [MW89]: $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{n,F}$. Then $L^2_{\mathrm{disc}}([G])$ is known, up to parametrizing cuspidal automorphic representations of $\mathrm{GL}_{d,F}$ for $d|n$, and turns out to be multiplicity free.

We can now close a gap. Namely, we never associated an automorphic representation to a newform. But with the material presented now we can give (at least two) formal definition. Let $f \in S_k$ be a newform. Then we can either associate to f the (irreducible) $\mathrm{GL}_2(\mathbb{A}_f) \times (\mathfrak{gl}_{2,\mathbb{C}}, \mathrm{O}_2(\mathbb{R}))$-submodule in $\mathcal{A}(\mathrm{GL}_2)$ generated by the automorphic form φ_f (cf. Section 3, or the irreducible, unitary subrepresentation of $L^2([\mathrm{GL}_2])$ generated by $\tilde{\varphi_f}$, cf. Section 3. Which construction is used has to be checked in the respective situation.

Satake parameters:

- Describe π_p if π is unramified at p, i.e.,

$$\pi_p^{\mathcal{G}_p(\mathbb{Z}_p)} \neq 0$$

with $\mathcal{G}_p \to \mathrm{Spec}(\mathbb{Z}_p)$ a reductive model of $G_{\mathbb{Q}_p}$ (which exists for almost all primes p).
- Thus, describe $\mathbb{C}$-algebra homomorphisms

$$\mathcal{H}(G(\mathbb{Q}_p), K) \to \mathbb{C}$$

for $K := \mathcal{G}_p(\mathbb{Z}_p)$.
- Langlands/Satake, cf. [GH19, Theorem 7.5.1., Corollary 7.5.2.], [BG10, Section 2.1.]: There exists
  - a "natural" algebraic group $\hat{G}$ over $\overline{\mathbb{Q}}$,
  - an automorphism $\mathrm{Fr}_p \colon \hat{G}(\overline{\mathbb{Q}}) \cong \hat{G}(\overline{\mathbb{Q}})$,

  - and a bijection

$$\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathcal{H}(G(\mathbb{Q}_p),K),\mathbb{C}) \overset{1:1}{\leftrightarrow}$$

$$\{\text{Frobenius semisimple } \hat{G}(\mathbb{C}) - \text{conjugacy classes in } \hat{G}(\mathbb{C}) \rtimes \mathrm{Fr}_p^{\mathbb{Z}}\}.$$

- The group $\hat{G}_{\mathbb{C}}$ has the dual root datum as G. Thus for example, cf. [Bor79]:
  - if $G = \mathrm{GL}_n$, then $\hat{G} = \mathrm{GL}_{n,\mathbb{C}}$,
  - if $G = \mathrm{SL}_n$, then $\hat{G} = \mathrm{PGL}_{n,\mathbb{C}}$,
  - if $G = \mathrm{SO}_{2n}$, then $\hat{G} = \mathrm{SO}_{2n,\mathbb{C}}$,
  - if $G = \mathrm{SO}_{2n+1}$, then $\hat{G} = \mathrm{Sp}_{2n,\mathbb{C}}$,
  - if $G = \mathrm{GSp}_{2n}$, then $\hat{G} = \mathrm{GSpin}_{2n+1,\mathbb{C}}$.
- If $G_{\mathbb{Q}_p}$ is split, then the automorphism Fr_p is trivial, and Frobenius semisimple conjugacy classes are just conjugacy classes of semisimple elements in $\hat{G}(\mathbb{C})$.
- If $G = \mathrm{GL}_n$, then semisimple conjugacy classes are uniquely determined by their characteristic polynomials.
- Let $\pi = \bigotimes'_p \pi_p \otimes \pi_\infty$ be an automorphic representation of G. Then we obtain the following analog of a system of Hecke eigenvalues:
  - a finite set S of primes, such that π_p is unramified for $p \notin S$,
  - for each $p \notin S$ a Frobenius semisimple conjugacy class $c_p(\pi)$ in $\hat{G}(\mathbb{C}) \rtimes \mathrm{Fr}_p^{\mathbb{Z}}$.
- For GL_n the eigenvalues of (each element in) $c_p(\pi)$ are the Satake parameters of π.
- Concretely, if the coset

$$\mathrm{GL}_n(\mathbb{Z}_p)\mathrm{Diag}(\underbrace{p,\ldots,p}_{i-\text{times}}, \underbrace{1,\ldots,1}_{(n-i)-\text{times}})\mathrm{GL}_n(\mathbb{Z}_p),$$

  has eigenvalue $\widetilde{a}_{p,i}$ on $\pi_p^{\mathrm{GL}_n(\mathbb{Z}_p)}$, then the elements in $c_p(\pi)$ have characteristic polynomial

$$X^n - p^{\frac{(1-n)}{2}}\widetilde{a}_{p,1}X^{n-1} + \ldots + (-1)^i p^{\frac{i(i-n)}{2}}\widetilde{a}_{p,i}X^i + \ldots + (-1)^n\widetilde{a}_{p,n},$$

  cf. [GH19, Section 7.2.].
- Even more concrete, for π generated by a newform $f = \sum\limits_{i=1}^{\infty} a_n q^n \in S_k(\Gamma_0(N),\chi)$ the $c_p(\pi), p \nmid N$, have characteristic polynomial

$$X^2 - p^{-1/2} p a_p X + \chi(p)p^k.$$

  Recall that $\pi \otimes_{\mathbb{C}} |\det|^{1/2}_{\text{adélic}}$ is L-algebraic. For $\pi \otimes_{\mathbb{C}} |\det|^{1/2}_{\text{adélic}}$ we obtain the polynomial

$$X^2 - a_p X + \chi(p)p^{k-1},$$

  which (after choosing an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$) was the characteristic polynomial of an arithmetic Frobenius at p.

The L-group:

- G reductive over $\mathbb{Q}$
- ℓ a prime.
- With a little work (cf. [GH19, Section 7.3.]) the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the reductive group $\hat{G}$ over $\overline{\mathbb{Q}}$ (the action is trivial if G is split).

- Set
$$^{L}G := \hat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$
- An L-parameter is by definition a $\hat{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy class of continuous homomorphisms
$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^{L}G,$$
whose projection to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the identity.
- If $F/\mathbb{Q}$ is finite and $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{n,F}$, then an L-parameter identifies with an isomorphism class of an n-dimensional ℓ-adic Galois representation of $\mathrm{Gal}(\overline{F}/F)$.

The Buzzard–Gee conjecture for L-algebraic automorphic representations:

- Let ℓ be a prime.
- Fix an isomorphism $\iota\colon \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$.
- Let π be an L-algebraic automorphic representation of G.
- Then Buzzard–Gee (cf. [BG10, Conjecture 3.2.2.]) conjecture that there exists an L-parameter
$$\rho_\pi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^{L}G$$
such that (in particular)
  - if $p \neq \ell$ is unramified for π, then each arithmetic Frobenius $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ maps to the conjugacy class of
  $$\iota(c_p(\pi)) \in \hat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{Frob}_p^{\mathbb{Z}}$$
  under
  $$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \hat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \twoheadrightarrow \hat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{Frob}_p^{\widehat{\mathbb{Z}}},$$
  - for each continuous representation ${}^{L}G \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$, which is algebraic on $\hat{G}(\overline{\mathbb{Q}}_\ell)$, the composition
  $$\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \xrightarrow{\rho} {}^{L}G \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$$
  is de Rham.
- This generalizes the association of Galois representations to (a twist of) the automorphic representation associated to some newform.
- Contrary to the case of GL_n the L-parameter ρ_π is *not* conjectured to be unique, cf. [BG10, Remark 3.2.4.].
- Moreover, we may chose the same L-parameter ρ for different L-algebraic automorphic representations π.
- For GL_n the Buzzard–Gee conjecture predicts in addition to the previous Langlands-Clozel-Fontaine-Mazur conjecture the existence of Galois representations attached to possibly non-cuspidal (L-algebraic) automorphic representations. These Galois representations are then no longer conjectured to be irreducible.

The Langlands group $\mathcal{L}$:

- Conjecturally, there should exist a (very big) locally compact group $\mathcal{L}$, the (global) Langlands group, with (at least) the following properties, cf. [GH19, Section 12.6.], [Art02], [LR87]:

  - There exists a canonical surjection
$$\mathcal{L} \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$
(in fact, $\mathcal{L} \twoheadrightarrow W_{\mathbb{Q}}$ with $W_{\mathbb{Q}}$ the global Weil group).
  - For each $n \geq 0$ the isomorphism classes $\mathrm{Irr}_n(\mathcal{L})$ of continuous irreducible n-dimensional $\mathbb{C}$-representations of $\mathcal{L}$ are naturally in bijection with the set Csp_n of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$.
  - Fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. Then there is an injection of the set $\mathcal{G}_{\mathrm{geom}}$ of isomorphism classes of irreducible, almost everywhere unramified ℓ-adic Galois representations $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which are de Rham at ℓ to the set Irr of isomorphism classes of irreducible representations of $\mathcal{L}$, with image corresponding to L-algebraic cuspidal automorphic representations.
  - For each place v of $\mathbb{Q}$ there is an injection
$$\mathcal{L}_v \to \mathcal{L}$$
of the local Langlands groups, which is well-defined up to conjugation in $\mathcal{L}$. Here:
    * $\mathcal{L}_v \cong W_{\mathbb{R}}$, the non-split extension of $\mathbb{C}^\times$ by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, if $v = \infty$.
    * $\mathcal{L}_v \cong W_{\mathbb{Q}_p} \times \mathrm{SU}_2(\mathbb{R})$ if $v = p$ prime and
$$W_{\mathbb{Q}_p} \cong I_{\mathbb{Q}_p} \rtimes \mathbb{Z} \subseteq \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \cong I_{\mathbb{Q}_p} \rtimes \hat{\mathbb{Z}}$$
is the subgroup of elements mapping to integral powers of Frobenius.
  - If $\pi \cong \bigotimes_v' \pi_v \in \mathrm{Csp}_n$ corresponds to a continuous representation $\rho_\pi \colon \mathcal{L} \to \mathrm{GL}_n(\mathbb{C})$, then π_v should correspond to $\rho_{\pi|\mathcal{L}_v}$ under the local Langlands correspondence for GL_n, cf. [GH19, Section 12.5.].
- If this conditions hold, then necessarily $\mathcal{L}^{\mathrm{ab}} \cong \mathbb{Q}^\times\backslash\mathbb{A}^\times$. Note that by global class field theory there exists a surjection $\mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})^{\mathrm{ab}}$.
- The existence of $\mathcal{L}$ is currently completely out of reach, even more conjecturally $\mathcal{L}$ should surject onto the "motivic Galois group" G_{mot} over $\mathbb{C}$, cf. [Art02], [LR87].

The global Langlands correspondence for general G (very, very rough):

- No precision is claimed, and all statements are close to being empty!
- For precise statements one should consult [GH19, Chapter 12] or [Art94].
- In order to parametrize non L-algebraic automorphic representations (at least those relevant for the decomposition of $L^2_{\mathrm{disc}}([G])$) one should replace the previous Galois version of L-parameters by (certain) L-parameters
$$\mathcal{L} \to {}^L G$$
(actually in the Weil form $W_{\mathbb{Q}} \ltimes \hat{G}(\mathbb{C})$ of the L-group).
- Then one hopes to construct a surjective map (cf. [GH19, Conjecture 12.6.2.])
$$\{(\text{certain}) \text{ automorphic representations}\} \xrightarrow{\mathrm{LL}} \{(\text{certain}) \ L-\text{parameters}\}.$$
- The fibers of LL are called L-packets. They are possibly infinite.
- A precise construction of a local Langlands correspondence (cf. [GH19, Conjecture 12.5.1.]) should yield a parametrization of the L-packets, cf. [GH19, (12.23)].

- For each $\pi \subseteq L^2_{\mathrm{disc}}([G])$ its multiplicity should be computable via the L-parameter $\mathrm{LL}(\pi)$ and the precise parametrizations of the L-packets, cf. [GH19, Conjecture 12.6.3.].
- However, for non-tempered representations the above should not be reasonable and one should consider Arthur parameters
$$\mathcal{L} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$
instead of L-parameters, cf. [Art94].

Important stuff, which was not mentioned in the lecture:

- L-functions, [Bor79], [GH19, Chapter 11, Section 12.7.],
- Functoriality, [GH19, Section 12.6.],
- The trace formula, [Art05], [GH19, Section 18],
- The Langlands program for function fields,
- ...

A glimpse on Shimura varieties:

- Let $K_\infty \subseteq G(\mathbb{R})$ be a maximal (connected) compact subgroup.
- Let $K \subseteq G(\mathbb{A}_f)$ be a (sufficiently small) compact-open subgroup.
- Recall that
  - $$[G] \to [G]/K = G(\mathbb{Q})A_G\backslash G(\mathbb{A})/K$$
    is a profinite covering of a real manifold
  - $$[G]/K \to X_K := [G]/KK_\infty = G(\mathbb{Q})A_G\backslash G(\mathbb{A})/KK_\infty$$
    is a K_∞-bundle over X_K, which is a disjoint union of arithmetic manifolds (=quotients of a symmetric spaces by arithmetic subgroups)
- Note that implicitly X_K depends on K_∞.
- Let us set (just to simplify some notations later)
$$\widehat{X} := \varprojlim_K X_K.$$

Stuff we encountered for $G = \mathrm{GL}_2$:

(1) The upper/lower halfplane $\mathbb{H}^\pm \cong G(\mathbb{R})/K_\infty$, which is naturally a *complex* manifold
(2) The holomorphic embedding
$$\mathbb{H}^\pm \to \mathbb{P}^1_{\mathbb{C}}$$
(3) The $G(\mathbb{R})$-equivariant vector bundles
$$\omega^k$$
on $\mathbb{H}^\pm$, or by pullback on X_K for $K \subseteq G(\mathbb{A}_f)$ compact-open.
(4) For $k \in \mathbb{Z}$ the space of modular forms
$$M_k \subseteq H^0(\widehat{X}, \omega^k),$$
where the RHS denotes *holomorphic* sections.
(5) The canonical compactification X_K^* of X_K, with the extension of ω^k on it.
(6) A scheme $\widehat{\mathcal{X}} \to \mathrm{Spec}(\mathbb{Q})$, whose $\mathbb{C}$-valued points are naturally isomorphic to $\widehat{X}$.

(7) The $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times G(\mathbb{A}_f)$-representation

$$\widetilde{H}^1_{\text{ét}}(\widetilde{\mathcal{X}}_{\overline{\mathbb{Q}}}, \mathbb{L})$$

associated to certain $G(\mathbb{A}_f)$-equivariant $\overline{\mathbb{Q}}_\ell$-local systems $\mathbb{L}$ on $\widetilde{\mathcal{X}}$, e.g., $\mathbb{L} \cong \overline{\mathbb{Q}}_\ell$.

(8) The Eichler–Shimura isomorphism relating S_k for $k \geq 2$, to certain

$$\widetilde{H}^1_{\text{ét}}(\widetilde{\mathcal{X}}_{\overline{\mathbb{Q}}}, \mathbb{L}).$$

(9) The Eichler–Shimura relation expressing a relation of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action and the $G(\mathbb{A}_f)$-action on

$$\widetilde{H}^1_{\text{ét}}(\widetilde{\mathcal{X}}_{\overline{\mathbb{Q}}}, \mathbb{L}),$$

which was a consequence of the mod p geometry of some modular curve.

(10) A map

$$\mathrm{LL}\colon \mathcal{A}_{\mathrm{mod}} \to \mathcal{G}_{\mathrm{mod}}$$

with (conjecturally) describable image.

For general G there are unfortunately no analogs - but wait: there exist G giving rise to Shimura varieties!

- Already (Item 1) fails for general G: The space

$$G(\mathbb{R})/K_\infty$$

need not be *complex* manifold! (E.g., if $G = \mathrm{GL}_3$, then the real dimension $\dim(\mathrm{GL}_3(\mathbb{R})/A_{\mathrm{GL}_3(\mathbb{R})}\mathrm{SO}_3(\mathbb{R})) = \frac{3(3+1)}{2} - 1 = 5$ is odd)
- In particular, everything related to holomorphicity has no analog for such G, e.g., Item 2, Item 3, Item 4, Item 6, Item 7,...
- That is bad news!
- Good news: For groups G giving rise to Shimura data (like $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F}$ for $F/\mathbb{Q}$ totally real, or GSp_{2n},... , cf. [Lan17], [Mil05], [Del71a], [Del79]), the real manifold

$$G(\mathbb{R})/K_\infty A_G$$

is a *complex* manifold.
- A Shimura datum is a pair (G, X) with G a reductive group G over $\mathbb{Q}$, and X a $G(\mathbb{R})$-conjugacy classes of morphisms $h\colon \mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ such that
  1) For each $h \in X$, the characters of $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acting on $\mathrm{Lie}(G_{\mathbb{R}})_{\mathbb{C}}$ (via $\mathrm{Ad} \circ h$) are either $z \mapsto \frac{z}{\overline{z}}$, $z \mapsto 1$, or $z \mapsto \frac{\overline{z}}{z}$.
  2) For each $h \in X$, the group $\{g \in G(\mathbb{C}) \mid h(i)gh(i)^{-1} = \overline{g}\}$ is compact modulo its center.
  3) The adjoint group G^{ad} has no factor, defined over $\mathbb{Q}$, for which the projection of $h \in X$ is trivial.
- E.g., take $G = \mathrm{GL}_2$, and X as the conjugacy class of the usual embedding $\mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{R})$. Note that $X \cong \mathbb{H}^\pm$.
- In general, for $h \in X$ the stabilizer K_h of h in $G(\mathbb{R})$ is compact modulo the center of $G(\mathbb{R})$ (this follows from condition 2)) and $X \cong G(\mathbb{R})/K_h(\cong G(\mathbb{R})/A_G K_\infty)$.

- For $K \subseteq G(\mathbb{A}_f)$ recall
$$X_K := G(\mathbb{Q})\backslash(G(\mathbb{A}_f)/K \times X).$$
Other notation: $\mathrm{Sh}_K(G, X)$ the Shimura variety of level K attached to (G, X).
- We get:
  (1) X is a disjoint union of hermitian symmetric domains, cf. [Mil05], [Del79].
  (2) To each $h \in X$ is naturally attached a parabolic subgroup $P_h \subseteq G(\mathbb{C})$ (cf. [CS17, Section 2.1]), and sending $h \to P_h$ defines an open embedding
  $$\pi\colon X \cong G(\mathbb{R})/K_h \to \mathcal{F}l \cong G(\mathbb{C})/P_h$$
  of X into a flag variety (this generalizes the embedding $\mathbb{H}^{\pm} \to \mathbb{P}^1(\mathbb{C})$).[16] In particular, X carries a natural complex structure. More is true (Baily-Borel): X_K is naturally a quasi-projective variety, cf. [Lan17, Theorem 2.4.1.].
  (3) The pullback of $G(\mathbb{C})$-equivariant vector bundles on $\mathcal{F}l$ gives rise to $G(\mathbb{Q})$-equivariant vector bundles on $G(\mathbb{A}_f) \times X$. By descent one one obtains automorphic vector bundles $\mathcal{E}$ on X_K for any $K \subseteq G(\mathbb{A}_f)$ compact-open, i.e., analogs of the $\omega^k, k \in \mathbb{Z}$, [Har88]. Note that $G(\mathbb{C})$-equivariant vector bundles on $\mathcal{F}l$ are equivalent to (algebraic) representations of P_h.
  (4), (5) If $\dim_{\mathbb{C}}(X_K) > 1$, there do exist compactifications X_K, but these are no longer canonical. However, it is possible to define subspaces in the cohomology of $H^*(X_K, \mathcal{E})$), which generalize M_k for $k \in \mathbb{Z}$, cf. [Har88].
  (6) The space $\widehat{X}$ (=inverse limit of complex manifolds) arises again as the $\mathbb{C}$-points of a scheme $\widehat{X} \to \mathrm{Spec}(E)$, the canonical model, defined over some number field (the "reflex field of the Shimura data"), cf. [Lan17, Theorem 2.4.3.].
  (8),(9),(10) Having the canonical model, one can attempt (sometimes sucessfully) to obtain analogs of points (18),(19),(20) and finally relate (certain) automorphic representations to (variants of) Galois representations, cf. [Har88], [BRb]. Needless to say that everything (compactifications, Eichler–Shimura isomorphism/relation, integral models, mod p geometry, interior vs intersection cohomology,...) is much more complicated!

[16] Viewing X as a "moduli space of (polarized) Hodge structures" clarifies this embedding, cf. [Del79].

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