LECTURE NOTES ON ÉTALE COHOMOLOGY

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Abstract. These notes are (hopefully regularly) updated lecture notes for a course on étale cohomology taught in WS 22/23 in Bonn. Please use with care. Any comments/questions/corrections are welcome![1]

Contents

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1. Introduction

The initial impetus on étale cohomology was André Weil's question on the existence of a cohomology theory for algebraic varieties in positive characteristic, which behaves as "the usual cohomoloy theory over $\mathbb{C}$". This aim was then established in tremendeous work of Grothendieck, Artin,... .

The aim for this course to explain the construction and properties of étale cohomology, and also what is actually understood with this "usual cohomology theory over $\mathbb{C}$". We will start with discussing the latter.

2. Singular homology for schemes?

In this section we want to adress the question on the possiblity to extend singular homology from topological spaces to schemes.

2.1. **Reminder on singular homology of topological spaces.** Let T be a topological space. Singular homology provides useful algebraic invariants of T. Let us recall its construction.

Definition 2.2. We define Δ as the category with objects $[n] := \{0, \ldots, n\}$ for each $n \geq 0$ and morphisms $\mathrm{Hom}_\Delta([n], [m])$ given by the set of monotone maps $f : [n] = \{0, \ldots, n\} \to [m] = \{0, \ldots m\}$, i.e., $i \leq j$ implies $f(i) \leq f(j)$.

More canonically, Δ could equivalently be defined as the category of finite, non-empty linearly ordered sets with order preserving maps.

The category Δ is the combinatorial analog of the collection of topological simplices

$$\Delta_n^{\mathrm{top}} := \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1,\ 0 \leq t_i \leq 1 \text{ for all } i\}$$

for $n \geq 0$.

To make this more precise we recall the useful definition of simplicial/cosimplical objects in a category.

Definition 2.3. Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a functor

$$\Delta^{\mathrm{op}} \to \mathcal{C},\ [n] \mapsto C_n,$$

and a cosimplicial object in $\mathcal{C}$ is a functor

$$\Delta \to \mathcal{C},\ [n] \mapsto C^n.$$

In both cases, morphisms are given by natural transformations of functor.

For example, a simplical set is a simplical object in $\mathcal{C} = (\mathrm{Sets})$, a simplicial topological space is a simplicial object in $\mathcal{C} = (\mathrm{Top})$, and so on. Intuitively, the value of a simplicial object on $[0]$ yields the "points" or "0-cells", the value on $[1]$ the "paths" or "1-cells", the two morphisms $[0] \to [1]$ in Δ map a 1-cell to either of its ends, and so on.

Now, the topological simplices $\Delta_n^{\mathrm{top}}, n \geq 0$, assemble into a cosimplicial topological space by sending $[n] \in \Delta$ to Δ_n^{top} and a monotone map $f : [n] \to [m]$ to the map

$$\Delta_n^{\mathrm{top}} \to \Delta_m^{\mathrm{top}},\ (t_0, \ldots, t_n) \mapsto (\sum_{j \in f^{-1}(0)} t_j, \ldots, \sum_{j \in f^{-1}(m)} t_j).$$

Most notably, for $i = 0, \ldots, n+1$ the injective monotone map $\delta_i : [n] \to [n+1]$ not hitting i defines an embedding of Δ_n^{top} into $\Delta_{n+1}^{\mathrm{top}}$ as a boundary simplex of codimension 1.

Now, we can define singular homology with coefficients in an arbitrary abelian group G.

Definition 2.4. Let T be a topological space.

(1) The singular complex $\mathrm{Sing}(T)$ of T is the simplicial set

$$\Delta^{\mathrm{op}} \to (\mathrm{Sets}),\ [n] \mapsto \mathrm{Hom}_{\mathrm{Top}}(\Delta_n^{\mathrm{top}}, T).$$

(2) The singular chain complex $C_\bullet^{\mathrm{sing}}(T; \mathbb{Z})$ of T is the complex of free abelian groups

$$\ldots \to \mathbb{Z}[\mathrm{Sing}(T)_n] \to \ldots \to \mathbb{Z}[\mathrm{Sing}(T)_1] \to \mathbb{Z}[\mathrm{Sing}(T)_0]$$

with differential $d = \sum_{i=0}^{n+1} (-1)^i d_i$, where $d_i : \mathrm{Sing}(T)_{n+1} \to \mathrm{Sing}(T)_n$ is the image under the functor $\mathrm{Sing}(T)$ of the injective monotone map $\delta_i : [n] \to [n+1]$ not hitting i.

(3) If G is an abelian group, then $C_\bullet^{\mathrm{sing}}(T; G) := G \otimes_{\mathbb{Z}} C_\bullet^{\mathrm{sing}}(T; \mathbb{Z})$.

(4) The n-the singular homology group $H_n(T; G)$ (with coefficients in the abelian group G) of T is by definition the n-th homology group $\mathcal{H}_n(C_\bullet^{\mathrm{sing}}(T; G))$ of the singular chain complex.

We leave it as an exercise to check that the properties of Δ imply that $d \circ d = 0$, i.e., that $C_\bullet^{\mathrm{sing}}(T; G)$ is a complex.[1] Given a complex $C_\bullet$ of abelian groups we denoted by $\mathcal{H}_n(C_\bullet)$ its n-homology group $\frac{\ker(d : C_n \to C_{n-1})}{\mathrm{im}(d : C_{n+1} \to C_n)}$.

Clearly, the constructions $T \mapsto \mathrm{Sing}(T)$, $T \mapsto C_\bullet^{\mathrm{sing}}(T; G)$, $T \mapsto H_n(T; G)$ are (covariantly) functorial in T and G.

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[1]Exercise: Let $\mathcal{A}$ be an additive category and $A_\bullet : \Delta \to \mathcal{A}$ a simplicial object. Define $d = \sum_{i=0}^{n+1} (-1)^i d_i : A_{n+1} \to A_n$ as in Definition 2.4. Then $d \circ d = 0$.

Although the simplicial set $\mathrm{Sing}(T)$ is in general ridiculously big, the excellent formal properties of singular homology make $H_*(T;G)$ a rather computable invariant. To state these formal properties one needs to define the relative singular chain complex for the pair (T,A) with A a topological subspace of T as

$$C_\bullet^{\mathrm{sing}}(T,A;G) := C_\bullet^{\mathrm{sing}}(T;G)/C_\bullet^{\mathrm{sing}}(A;G)$$

and the relative singular homology groups as

$$H_n(T,A;G) := \mathcal{H}_n(C_\bullet^{\mathrm{sing}}(T,A;G)).$$

Now the good formal properties of singular homology are summarised in the following theorem.

Theorem 2.5. (1) *Singular homology is homotopy invariant, i.e., for any topological space T the projection $T \times \Delta_1^{\mathrm{top}} \to T$ induces isomorphisms*

$$H_n(T \times \Delta_1^{\mathrm{top}};G) \to H_n(T;G)$$

for any $n \geq 0$.

(2) *For a pair (T,A) with A a subspace of T there exists a natural long exact sequence*

$$\ldots \to H_{n+1}(A;G) \to H_{n+1}(T;G) \to H_{n+1}(T,A;G) \to H_n(A;G) \to H_n(T;G) \to \ldots.$$

(3) *We have $H_0(\{*\};G) = G$ and $H_n(\{*\};G) = 0$ for $n > 0$.*

(4) *Singular homology satisfies excision, i.e., given a pair (T,A) and a subspace $Z \subseteq A$ whose closure is contained in the interior of A, then the inclusion $(T-Z, A-Z) \to (T,A)$ induces isomorphisms*

$$H_n(T-Z, A-Z;G) \to H_n(T,A;G)$$

for all $n \geq 0$.

Proof. The homotopy invariance is proven in [17, Theorem 2.10]. The existence of the natural long exact sequence follows from the definition and the long exact sequence associated to a short exact sequence of complexes.If $T = \{*\}$ is a one-point space, then $\mathrm{Sing}(T)$ is the constant simplicial object with value $\{*\}$. Then $C_\bullet^{\mathrm{sing}}(T;G)$ is given by the complex

$$\ldots \xrightarrow{0} G \xrightarrow{1} G \xrightarrow{0} G \to 0 \to \ldots,$$

and the computation follows. Excision is proven in [17, Theorem 2.20]. □

From here we can draw the following consequences.

Proposition 2.6. (1) *If (T,A) is a good pair, i.e., A is a deformation retract of some open neighborhood of it, then*

$$H_n(T,A;G) \cong H_n(T/A, A/A;G)$$

for all $n \geq 0$. In particular, this holds if T is a CW-complex and $A \subseteq T$ a subcomplex.

(2) *If S^n denotes the n-dimensional sphere, then $H_n(S^n;G) = \mathbb{Z} = H_0(S^n;G)$ and $H_i(S^n;G) = 0$ for $i \neq 0, n$.*

Proof. The first point is [17, Proposition 2.22.]. For the second, see [17, Example 2.23]. □

Example 2.7. As an example for how to use the properties of singular homology, we compute the singular homology of

$$\mathbb{C}\mathrm{P}^n := \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\mathbb{C}^\times} \cong \frac{S^{2n+1}}{S^1}.$$

We use homogeneous coordinates $x = (x_0 : \ldots : x_n) \in \mathbb{C}\mathrm{P}^n$ to represent elements in $\mathbb{C}\mathrm{P}^n$. The inclusion

$$\mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n,\ (x_0 : \ldots : x_{n-1}) \to (x_0 : \ldots : x_{n-1} : 0)$$

defines a good pair as the open neighborhood $U := \mathbb{C}\mathrm{P}^n \setminus \{(0 : \ldots : 0 : 1)\}$ of $\mathbb{C}\mathrm{P}^{n-1}$ identifies with the topological space of a complex line bundle on $\mathbb{C}\mathrm{P}^{n-1}$. By Proposition 2.6 we can conclude

$$H_*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^{n-1};\mathbb{Z}) \cong H_*(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1}, \{*\};\mathbb{Z}).$$

Now, we claim that $\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1} \cong S^{2n}$. Granting this the calculation can be finished by considering induction on n the long exact sequence

$$\ldots \to H_i(\mathbb{C}\mathrm{P}^{n-1};\mathbb{Z}) \to H_i(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to H_i(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1}, \{*\};\mathbb{Z}) \to \ldots$$

with the result that $H_i(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ for $i \leq 2n$ even, and zero otherwise. Let D^{2n} be the closed $2n$-dimensional unit disc. Then we have a continuous map

$$D^{2n} \to S^{2n+1},\ w \mapsto (w, \sqrt{1-|w|^2}),$$

which maps the boundary ∂D^{2n} to $\mathbb{C}\mathrm{P}^{n-1}$. It is easy to see that the resulting map

$$S^{2n} \cong D^{2n}/\partial D^{2n} \to \mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1}$$

is a homeomorphism. Indeed, given a point $y = (y_0, \ldots, y_n) \in S^{2n+1} \subseteq \mathbb{C}^{n+1}$ there exists a unique element $z \in S^1$ such that $z \cdot y_n \in \mathbb{R}_{>0}$, from which the statement follows easily.

Let us give another example showing the usefulness of excision.

Example 2.8. Let T be a topological space and $Z \subseteq T$ a closed subspace. If $U \subseteq T$ is an open neighborhood of Z, then excision to the $T \setminus U$ is a closed subset contained in the open subset $T \setminus Z$. Then implies that the map $(U, U \setminus Z) \to (T, T \setminus Z)$ induces an isomorphism

$$H_*(U, U \setminus Z; G) \to H_*(T, T \setminus Z; G),$$

i.e., the groups $H_*(T, T \setminus Z; G)$ only depend on an open neighborhood of Z. If $Z = \{t\}$ with $t \in T$ a (closed) point, then $H_*(T, T \setminus \{t\}; G)$ is called the local homology at t and can be seen as a measure of how "singular" t is. For example, if T is a manifold of dimension n, then

$$H_*(T, T \setminus \{t\}; G) \cong H_*(D^n, D^n \setminus \{0\}; G)$$

and as $D^n \setminus \{0\}$ is homotopy equivalent to S^{n-1} we can conclude (using the long exact sequence in singular homology)

$$H_i(T, T \setminus \{t\}; G) \cong G$$

if $i = n$ and 0 if $i \neq n$. As local homology is invariant under homeomorphism (but not in general under homotopy equivalences), we can conclude that two manifolds can be homeomorphic only if their dimensions agree.

On the other hand, if $T = CS := [0,1] \times S/0 \times S$ is the cone of another (connected) topological space S and t the vertex, then T is contractible and hence

$$H_n(T, T \setminus \{t\}; G) \cong H_{n-1}(T \setminus \{t\}; G) \cong H_{n-1}(S; G)$$

can be rather arbitrary. In particular, cones are rarely homeomorphic to manifolds.

2.9. **Attempts to transport singular homology to schemes.** A first attempt to implement a homology theory to schemes might be to look at the singular homology of the underlying topological space. But in nearly all cases of interest this does not yield substantial information. The reason is the particularity of the underlying topological spaces of schemes.

Lemma 2.10. *Let T be a topological space with a generic point η, i.e., $\overline{\{\eta\}} = T$. Then T is contractible.*[2]

Proof. We claim that there exists a homotopy $H \colon T \times [0,1] \to T$ with $H_{|T\times 0}$ the identity and $H_{|T\times 1}$ the constant map with value η. Indeed, define $H(t,x) = t$ if $x = 0$ and $H(t,x) = \eta$ otherwise. Then H is continuous. Indeed, if $U \subseteq T$ is open and non-empty, then $\eta \in U$ and thus $H^{-1}(U) = U \times [0,1] \cup T \times (0,1]$ is open. □

The schemes of most interest in algebraic geometry are schemes of finite type over a field, and being noetherian schemes, their underlying topological spaces admit a decomposition into irreducible components. Thus, if X is of finite type over a field k, then the only information encoded in

$$H_*(|X|; \mathbb{Z})$$

is given by the configuration of the irreducible components $T_1, \ldots, T_r$ of $|X|$ (and the irreducible components of their iterated intersections). Heuristically, we see that there are just not enough "interesting maps"

$$\Delta_n^{\mathrm{top}} \to |X|$$

from topological n-simplices to topological spaces underlying schemes. A less naive attempt could therefore be to replace Δ_n^{top} by the algebraic n-simplex

$$\Delta_n^{\mathrm{alg}} := \{(t_0, \ldots, t_n) \in \mathbb{A}_k^{n+1} \mid \sum_{i=0}^{n} t_i = 0\}.$$

(for discussion that follows we fix a base field k). Using the same formula as for the topological n-simplices we get a cosimplicial scheme $\Delta_\bullet^{\mathrm{alg}}$, and given any scheme X over k we can form the "(naive) algebraic singular complex" $\mathrm{Sing}^{\mathrm{alg}}(X)$

$$[n] \mapsto \mathrm{Hom}_k(\Delta_n^{\mathrm{alg}}, X).$$

[2]This statement with proof was suggested by Louis Jaburi.

From here we can now speak about (naive) algebraic singular chain $C_\bullet^{\mathrm{naive}}(X;\mathbb{Z})$

$$\ldots \to \mathbb{Z}[\mathrm{Sing}^{\mathrm{alg}}(X)_1] \to \mathbb{Z}[\mathrm{Sing}^{\mathrm{alg}}(X)_0]$$

as for singular homology. The adjective "naive" is put on purpose as the next example shows.

Example 2.11. Assume that X/k is a proper, smooth curve of genus > 0. We claim that any map $f\colon \Delta_n^{\mathrm{alg}} \cong \mathbb{A}_k^n \to X$ of schemes over k is constant with value a k-rational point. Namely, any constant map must have image a k-rational point (by evaluating on a k-rational point on $\mathbb{A}_k^n$) and constancy may be checked over an algebraically closed field. Hence, we may assume that k is algebraically closed. Now, if $n = 1$, then $f\colon \mathbb{A}_k^1 \to X$ extends to a map $\mathbb{P}_k^1 \to X$ and must therefore be constant as the genus of X is > 0.[3] If n is arbitrary, then f is constant, when restricted to each line in $\mathbb{A}_k^n$ and thus must be constant.[4] Hence, the simplicial set $\mathrm{Sing}^{\mathrm{alg}}(X)$ is actually constant with value $X(k)$. Thus, $H_\bullet^{\mathrm{naive}}(X;\mathbb{Z}) = \mathbb{Z}[X(k)]$.

Thus, this attempt does not yield something useful. It is maybe even worse than our previous attempt as for a reducible curve with components of higher genus we lost the information on the configuration (note that Δ_n^{alg} is irreducible, and hence map into a single irreducible component).

It is a bit fascinating that a slight variant of this naive construction works. The following won't be necessary for the rest of cours and is just included for curiosity. In fact, the proof of Theorem 2.12 uses techniques that we will touch during this cours (and others).

We saw above that in general there are not "enough" maps $\Delta_n^{\mathrm{alg}} \to X$. Suslin had the idea to solve this issue by defining the Suslin complex associated with X as the complex $C_\bullet^{\mathrm{Sus}}(X)$ with terms given by

$$C_n^{\mathrm{Sus}}(X) := \mathbb{Z}[\{Z \subseteq \Delta_n^{\mathrm{alg}} \times_{\mathrm{Spec}(k)} X \mid Z \text{ integral and } Z \to \Delta_n^{\mathrm{alg}} \text{ finite and surjective}\}]$$

and simplicial structure maps induced by the pullbacks of cycles. [5] For any abelian group G the homology

$$H_*^{\mathrm{Sus}}(X;G) := \mathcal{H}_*(C_*^{\mathrm{Sus}}(X) \otimes_{\mathbb{Z}} G)$$

is called the Suslin homology of X (over k, with coefficients in G).

Theorem 2.12 (Suslin, Voevodsky)**.** *If $k = \mathbb{C}$, $X \to \mathrm{Spec}(k)$ is separated of finite type, and $n \in \mathbb{Z}$ non-zero, there exist a natural isomorphism*

$$H_*^{\mathrm{Sus}}(X;\mathbb{Z}/n) \cong H_*^{\mathrm{sing}}(X(\mathbb{C});\mathbb{Z}/n)$$

between mod n Suslin homology of X and the mod n singular homology of the analytification $X(\mathbb{C})$ of X.

Proof. [28, Theorem 8.3] □

Thus, Suslin homology does give the "right answer" over $\mathbb{C}$. As Suslin homology is defined by algebraic cycles, it is difficult to work with, e.g., excision does not work as nicely for schemes as for topological spaces. Although it is still homotopy invariant in the sense that the projection $X \times_{\mathrm{Spec}(k)} \mathbb{A}_k^1 \to X$ induces an isomorphism $H_*^{\mathrm{Sus}}(X \times_{\mathrm{Spec}(k)} \mathbb{A}_k^1; G) \to H_*^{\mathrm{Sus}}(X;G)$.

We will speak about analytification in more detail soon. For the moment let us just say that if $X \subseteq \mathbb{A}_{\mathbb{C}}^n$ is locally closed, then its analytification $X(\mathbb{C}) \subseteq \mathbb{A}_{\mathbb{C}}^n(\mathbb{C}) = \mathbb{C}^n$ is given the subspace topology for the product topology on $\mathbb{C}^n$ coming from the usual metric topology on $\mathbb{C}$. Similarly, for $X \subseteq \mathbb{P}_{\mathbb{C}}^n$ locally closed.

Theorem 2.12 fails without passing to mod n coefficients, cf. [28, Theorem 3.1]: $H_1^{\mathrm{Sus}}(X;\mathbb{Z}) = 0$ for an affine smooth curve over k, but in general $H_1^{\mathrm{sing}}(X(\mathbb{C});\mathbb{Z}) \neq 0$ (e.g., $X = \mathrm{Spec}(k[t,t^{-1}])$). If $X = \mathrm{Spec}(k[t,t^{-1}])$, then they prove $H_0^{\mathrm{Sus}}(X,\mathbb{Z}) \cong k^\times \oplus \mathbb{Z}$.

2.13. **Singular cohomology and sheaf cohomology.** So far, the outlined attempts of transporting "usual" homology theory to schemes failed or weren't too useful. But well, Weil did not ask for a *homology* theory, but a *cohomology* theory and this for a good reason.

Let us first recall the definition of singular cohomology.

[3]This is, for example, an instance of Lüroth's theorem: each non-trivial subextension of $k(t)$ is isomorphic to $k(s)$.

[4]More precisely, given a line L let $x_L \in X(k)$ be its constant value. As each two lines can be connected via some chain of lines, the point $x := x_L$ is independent of L. As X is separated, the locus where f and the constant map with value x agree is closed. As k is algebraically closed, this locus must be $\mathbb{A}_k^n$ as it contains each line in $\mathbb{A}_k^n$.

[5]This requires some work to make precise and we skip this here.

Definition 2.14. Let T be a topological space and G an abelian group. Then we define the singular cochain complex

$$C^\bullet_{\mathrm{sing}}(T;G) := \mathrm{Hom}_{\mathbb{Z}}(C^{\mathrm{sing}}_\bullet(T;\mathbb{Z}), G),$$

whose cohomology $H^\bullet_{\mathrm{sing}}(T;G) := \mathcal{H}^n(C^\bullet_{\mathrm{sing}}(T;G))$ is by definition the singular cohomology of X (with coefficients in G).

Thus, for $n \geq 0$ the abelian group $C^n_{\mathrm{sing}}(T;G)$ identifies with the abelian group of maps (of sets) $\mathrm{Sing}(T)_n \to G$.

Clearly, singular cohomology is a contravariant functor in T (and a covariant functor in G). Given a subspace $A \subseteq T$ we can define the relative singular cochain complex

$$C^\bullet_{\mathrm{sing}}(T, A; G) := \ker(C^\bullet_{\mathrm{sing}}(T;G) \to C^\bullet_{\mathrm{sing}}(A;G)).$$

Then singular cohomology satisfies natural analogs of Theorem 2.5, Proposition 2.6 making it as useful as singular homology.

Remark 2.15. Singular cohomology and singular homology are "dual" to each other. Heuristically, this follows from the definition. The perhaps cleanest way to express this duality is via passage to the derived category $\mathcal{D}(\mathbb{Z})$ of abelian groups.[6] Then

$$R\mathrm{Hom}_{\mathbb{Z}}(C^{\mathrm{sing}}_\bullet(T;\mathbb{Z}), \mathbb{Z}) \cong C^\bullet_{\mathrm{sing}}(T;\mathbb{Z})$$

(essentially) by definition. If $H^{\mathrm{sing}}_i(T;\mathbb{Z})$ is finitely generated for each i (e.g., if T is a compact manifold or a finite CW complex), then[7]

$$R\mathrm{Hom}_{\mathbb{Z}}(C^\bullet_{\mathrm{sing}}(T;\mathbb{Z}), \mathbb{Z}) \cong C^{\mathrm{sing}}_\bullet(T;\mathbb{Z}),$$

i.e., singular cohomology determines singular homology in this case.

The real advantage of singular *co*homology over singular homology is its comparison to sheaf cohomology.

Let us recall the construction of sheaf cohomology from last semester.[8]

Let T be a topological space and $\mathcal{F}$ a sheaf of abelian groups on T. Then the i-th sheaf cohomology group $H^i(T, \mathcal{F})$ of T is by definition the value on $\mathcal{F}$ of the i-th right derived functor of the global section functor $\Gamma(T, -)$. This can be computed as follows:

Take any resolution

$$0 \to \mathcal{F} \to I^0 \to I^1 \to I^2 \to \dots$$

by acyclic sheaves I^j (e.g., I^j could be an injective sheaf, or flasque[9], or...). Then apply $\Gamma(T, -)$ to $I^\bullet$ and take cohomology groups

$$H^i(T, \mathcal{F}) \cong \mathcal{H}^i(\Gamma(T, I^\bullet)).$$

The object $R\Gamma(T, \mathcal{F}) := \Gamma(T, I^\bullet) \in \mathcal{D}(\mathbb{Z})$ is the total derived functor of $\Gamma(T, -)$ (evaluated on $\mathcal{F}$). For the moment, we will only be interested in the case that $\mathcal{F} = \underline{G}$ is the constant sheaf associated to some abelian group G, i.e., for $U \subseteq T$ open $\underline{G}(U) := \mathrm{Hom}_{\mathrm{cont}}(U, G)$ with G given the discrete topology.

More generally, for any continuous map $f\colon T \to S$ we set $Rf_*(\mathcal{F}) := f_*(I^\bullet)$ as an object in the derived category $\mathcal{D}(S, \underline{\mathbb{Z}})$ of sheaves of abelian groups on S. If $g\colon S \to W$ is continuous, then $Rg_* \circ Rf_* \cong R(g \circ f)_* \colon \mathcal{D}(T, \underline{\mathbb{Z}}) \to \mathcal{D}(W, \underline{\mathbb{Z}})$ (when the functors are extended to the derived category). If $S = \{*\}$, then $\mathcal{D}(S, \underline{\mathbb{Z}}) \cong \mathcal{D}(\mathbb{Z})$ and $Rf_* \cong R\Gamma(T, -)$.

We can now prove the comparison of singular cohomology and sheaf cohomology.

Theorem 2.16. *Assume that T is a locally contractible[10] topological space, e.g., a real manifold. For any abelian group G there exists a natural (in T and G) isomorphism*

$$H^\bullet_{\mathrm{sing}}(T;G) \cong H^\bullet(T, \underline{G}).$$

[6] A reminder on derived categories will come during the course. For the moment, we only recall that $\mathcal{D}(\mathbb{Z})$ is the localization of the category of complexes of abelian groups at the class of quasi-isomorphisms, i.e., at those morphisms of complexes inducing isomorphisms on each cohomology group.

[7] Exercise: If $K \in \mathcal{D}(\mathbb{Z})$ is an object and $\mathcal{H}^i(K)$ finitely generated for each $i \in \mathbb{Z}$, then the natural map $K \to R\mathrm{Hom}_{\mathbb{Z}}(R\mathrm{Hom}_{\mathbb{Z}}(K, \mathbb{Z}), \mathbb{Z})$ is an isomorphism. (Hint: Non-canonically, $K \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i(K)$.)

[8] A more detailed reminder on cohomology will have to appear later in the course.

[9] A sheaf $\mathcal{G}$ is flasque if for all opens $U \subseteq V$ the restriction $\mathcal{G}(V) \to \mathcal{G}(U)$ is surjective.

[10] This means that for each point $t \in T$ and open neighborhood U of t there exists a contractible open neighborhood V of t contained in U.

The proof is taken from [29, Theorem 4.47] and only works under the mild assumption that T is Hausdorff and any open set U of T is paracompact.[11] As the proof will show we even get an isomorphism

$$C^\bullet_{\mathrm{sing}}(T;G) \cong R\Gamma(T,\underline{G}) \in \mathcal{D}(\mathbb{Z}).$$

Proof. We can define a presheaf $\mathcal{F}^\bullet$ of complexes on T by sending an open $U \subseteq T$ to its complex $C^\bullet_{\mathrm{sing}}(U;G)$ of singular cochains (with values in G). Sheafifying this presheaf of complexes yields a complex

$$\mathcal{C}^0_{\mathrm{sing},G} \to \mathcal{C}^1_{\mathrm{sing},G} \to \dots$$

of sheaves on T. Note that by our assumption that T is locally contractible this complex is a resolution of $\underline{G}$ because for $U \subseteq T$ a contractible open subset we have $H^i_{\mathrm{sing}}(U;G) \cong G$ if $i = 0$ and $H^i_{\mathrm{sing}}(U;G) \cong 0$ if $i \neq 0$. It suffices to check two statements:

(1) The sheaves $\mathcal{C}^i_{\mathrm{sing},G}$ are flasque, and in particular, $\Gamma(T,\mathcal{C}^\bullet_{\mathrm{sing},G})$ calculates $R\Gamma(T,\underline{G})$.
(2) The natural map $C^\bullet_{\mathrm{sing}}(T;G) \to \Gamma(T,\mathcal{C}^\bullet_{\mathrm{sing},G})$ is a quasi-isomorphism.

Given an open $U \subseteq T$ with an open covering $U = \bigcup\limits_{i\in I} U_i$ the sequence

$$C^n_{\mathrm{sing}}(U;G) \to \prod_{i\in I} C^n_{\mathrm{sing}}(U_i;G) \to \prod_{i,j\in I} C^n_{\mathrm{sing}}(U_i \cap U_j;G)$$

is exact in the middle because a compatible system of cochains on the U_i can be induced by a cochain on X just by setting the value to zero on any map $\Delta^{\mathrm{top}}_n \to U$, which does not factor through any U_i. This implies that $\mathcal{F}^n(U) = C^n_{\mathrm{sing}}(U;G) \to \mathcal{C}^n_{\mathrm{sing},G}(U)$ is surjective for any U under the assumption that each open in T is paracompact and T is Hausdorff.[12] However, the left map of the above sequence need not be injective as cochains on U could take non-zero values only on those $\Delta^{\mathrm{top}}_n \to U$ not factoring through some U_i. We can conclude that

$$\mathcal{C}^n_{\mathrm{sing},G}(U)$$

is the quotient of $C^n_{\mathrm{sing}}(U;G)$ by the subgroup $C^n_{\mathrm{sing}}(U;G)_0$ of cochains which restrict to 0 on some open cover of U. As $C^n_{\mathrm{sing}}(T;G) \to C^n_{\mathrm{sing}}(U;G)$ is surjective for any open $U \subseteq T$, we can conclude that the sheaves $\mathcal{C}^n_{\mathrm{sing},G}$ are flasque. To see the second claim, let $U \subseteq T$ be an open set with open covering $\mathcal{U} := \{U_i\}_{i\in I}$. Define $C^{\mathrm{sing}}_\bullet(U;\mathbb{Z})_{\mathcal{U}} \subseteq C^{\mathrm{sing}}_\bullet(U;\mathbb{Z})$ be the subcomplex spanned by the simplices $\Delta^{\mathrm{top}}_n \to U$, which factor through some U_i. By the theorem of small simplices, [17, Proposition 2.21], the inclusion $C^{\mathrm{sing}}_\bullet(U;\mathbb{Z})_{\mathcal{U}} \to C^{\mathrm{sing}}_\bullet(U;\mathbb{Z})$ is a homotopy equivalence of chain complexes. This implies (by applying $\mathrm{Hom}_{\mathbb{Z}}(-,G)$) that

$$C^\bullet_{\mathrm{sing}}(U;G) \to \mathrm{Hom}_{\mathbb{Z}}(C^{\mathrm{sing}}_\bullet(U;\mathbb{Z}),G)$$

is a homotopy equivalence. Now the kernel $K_{\mathcal{U}}$ of this level wise surjection is exactly the subcomplex of singular cochains whose restriction to U_i vanishes for each $i \in I$. We can conclude that $K_{\mathcal{U}}$ is acyclic, i.e., $\mathcal{H}^i(K_{\mathcal{U}}) = 0$ for all $i \in \mathbb{Z}$. From here we can now conclude (2) and finish the proof. □

Let us recall that sheaf cohomology with coefficients in $\underline{G}$ is contravariant in G. Namely, let $f\colon T \to S$ be a map of topological spaces and write $\underline{G}_T, \underline{G}_S$ for the respective constant sheaves with value G. Then $f^{-1}\underline{G}_S = \underline{G}_T$ and there is a natural map $\underline{G}_S \to Rf_*f^{-1}\underline{G}_S \cong Rf_*(\underline{G}_T)$. As $R\Gamma(S,Rf_*(-)) \cong R\Gamma(T,-)$ we get a natural map $R\Gamma(S,\underline{G}_S) \to R\Gamma(T,\underline{G}_T)$. Passing to cohomology defines the pullback $H^*(S,\underline{G}_S) \to H^*(T,\underline{G}_T)$. If T,S are locally contractible, and $\underline{G}_T \cong \mathcal{C}^\bullet_{\mathrm{sing},G,T}$ the quasi-isomorphism constructed in Theorem 2.16 for T, then $Rf_*(\underline{G}_T) \cong$

[11] The general case is treated in [26] or [23]. We thank Sven Manthe for pointing out this issue to us.

[12] Set $\mathcal{F} := \mathcal{F}^n$ with sheafification $\mathcal{G} := \mathcal{C}^n_{\mathrm{sing},G}$. Let $\theta\colon \mathcal{F} \to \mathcal{G}$ be the natural map of presheaves. Let $s \in \mathcal{G}(U)$. Then there exists a locally finite cover $U = \bigcup\limits_{i\in I} U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that $\theta(s_i) = s_{|U_i}$. Dieudonné's theorem implies that the paracompact, Hausdorff space U is normal, i.e., any two disjoint closed subsets admit disjoint open neighborhoods. This implies that we may assume that there exists an open cover $T = \bigcup\limits_{i\in I} W_i$ such that $\overline{W_i} \subseteq U_i$. For $t \in T$ choose now a neighborhood V_t such that $I_t := \{i \in I \mid V \cap W_i \neq \emptyset\}$ is finite. If $i \in I_t$ and $x \notin \overline{W_i}$, then we may replace V_t by the open neighborhood $V_t \setminus \overline{W_i}$ of t. As I_t is finite, we may thus assume that for $i \in I_t$ we get $t \in \overline{W_i} \subseteq U_i$. Now replace V_t by $V_t \cap \bigcap_{i\in I_t} U_i$. Then V_t is open and still $t \in V_t$. Now we get that $V_t \cap W_i \neq \emptyset \Rightarrow V_t \subseteq U_i$ for all $i \in I$ (but $V_t \cap U_i \neq \emptyset$ does not imply that $V_t \subseteq U_i$, for this reason introducing the W_i was important). Shrinking V_t further we may assume that $s_t := s_{i|V_t} \in \mathcal{F}(V_t)$ is independent of $i \in I_t$. Now let $t, r \in U$ be two points and let $z \in V_t \cap V_r$. If $z \in W_j$ for some $j \in I$. Then $W_j \cap V_t \cap V_r \neq 0$, which implies that $V_t \cup V_r \subseteq U_j$. This implies that

$$s_{t|V_t\cap V_r} = s_{j|V_t\cap V_r} = s_{r|V_t\cap V_r}.$$

By the proven properties of $\mathcal{F}$ the family $\{s_t \in \mathcal{F}(V_t)\}_{t\in U}$ can therefore be lifted to some section $a \in \mathcal{F}(U)$. By construction, $\theta(a) = s$ as desired.

$f_*(\mathcal{C}^\bullet_{\mathrm{sing},G,T})$ because the $\mathcal{C}^\bullet_{\mathrm{sing},G}$ are flasque. The pullback in singular cohomology defines a morphism $\mathcal{C}^\bullet_{\mathrm{sing},G,S} \to f_*(\mathcal{C}^\bullet_{\mathrm{sing},G,T})$, which realizes the morphism $\underline{G}_S \to Rf_*(\underline{G_T})$ via a morphism between complexes of flasque sheaves[13]. Using the natural quasi-isomorphisms $C^\bullet_{\mathrm{sing}}(S,G) \cong \Gamma(S, \mathcal{C}^\bullet_{\mathrm{sing},G,S})$, $C^\bullet_{\mathrm{sing}}(T,G) \cong \Gamma(S, f_*(\mathcal{C}^\bullet_{\mathrm{sing},G,T})) \cong \Gamma(T, \mathcal{C}^\bullet_{\mathrm{sing},G,T})$ we can conclude that the isomorphism in Theorem 2.16 is indeed natural in T. The naturality in G is clear.

In general, singular and sheaf cohomology disagree. Namely, $H^0_{\mathrm{sing}}(T;\mathbb{Z})$ identifies with $\mathbb{Z}$-valued maps on the set of path-connected components of T, while $H^0(T;\underline{\mathbb{Z}})$ identifies with the set of continuous homomorphisms $T \to \mathbb{Z}$, i.e., with continuous maps from the quotient topological space of connected components of T.

Last term we applied sheaf cohomology for *quasi-coherent* sheaves on schemes with some success, but for *constant* coefficients sheaf cohomology on topological spaces underlying schemes doesn't yield much.

Lemma 2.17. *If T is an irreducible topological space, i.e., T is non-empty and $T = Z_1 \cup Z_2$ with $Z_1, Z_2 \subseteq T$ closed implies $T = Z_i$ for some $i = 1, 2$, and G an abelian group, then the constant sheaf $\underline{G}$ on X is flasque. In particular, $H^i(T, \underline{G}) = 0$ for $i > 0$.*

Proof. The irreducibility of T implies that each non-empty open set of T is connected, in fact irreducible. This implies that for each inclusion $V \subseteq U$ of non-empty open sets in T the map $\underline{G}(U) = G \to \underline{G}(V) = G$ (the identity) is surjective. This implies that $\underline{G}$ is flasque. □

Still sheaf cohomology of topological spaces can yield interesting invariants for schemes, if the schemes are of locally finite type over $\mathbb{C}$. Before discussing this further we will spend some time on settling the process of "analytification".

3. Analytification of schemes

We now discuss in some details analytification of schemes, which are locally of finite type over $\mathbb{C}$, following the (classic) references [27] and [14].

3.1. Properties of locally ringed spaces and morphisms of locally ringed spaces.

Many notions familiar from scheme theory generalize easily to locally ringed spaces or to morphisms of locally ringed spaces. We spell out some of the details, cf. [15].

Definition 3.2. Let $(X, \mathcal{O}_X)$ be a locally ringed space. Then we call X normal at x (or regular, reduced, Cohen-Macauley, of dimension n,...) if the local ring $\mathcal{O}_{X,x}$ is normal (or regular, reduced, Cohen-Macauley, of Krull dimension n,...). If X is normal (or regular, ...) at each of its points, then we call X normal (or regular, ...).

We can extend several notions to notions for morphisms of locally ringed spaces.

Definition 3.3. Let $f\colon Y \to X$ be a morphism of locally ringed spaces.

1. f is called flat if for all $y \in Y$ the map $f^\sharp\colon \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is flat.
2. f is called an immersion if $|f|\colon |Y| \to |X|$ identifies $|Y|$ with a subtopological space of $|X|$, and for all $y \in Y$ the map $f^\sharp\colon \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is surjective.
3. f is a closed immersion if f is an immersion and $f(Y) \subseteq X$ is closed.
4. f is an open immersion if $f(Y) \subseteq X$ is open and for all $y \in Y$ the map $f^\sharp\colon \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is an isomorphism. (Equivalently: $f(Y)$ is open and f induces an isomorphism $(Y, \mathcal{O}_Y) \cong (f(Y), \mathcal{O}_{X|f(Y)})$.)

Example 3.4. We recall that the category of locally ringed spaces admits all finite limits, i.e., it has fiber products, cf. [19]. Given a morphism $f\colon Y \to X$ of locally ringed spaces we can therefore form the diagonal $\Delta_f\colon Y \to Y \times_X Y$, which is always an immersion.

Example 3.5. If $f\colon Y \to X$ is a (closed, open) immersion and $Z \to X$ any morphism of locally ringed spaces, then the base change $f'\colon Y \times_X Z \to Z$ of f is again a (closed, open) immersion.

Example 3.6. Let $(X, \mathcal{O}_X)$ be a locally ringed space. Closed immersions (up to isomorphisms, and in the sense of Definition 3.3) are in bijection with ideal sheaves $\mathcal{I} \subseteq \mathcal{O}_X$, similar to the case of schemes. Given a closed immersion $f\colon Y \to X$ one gets an ideal sheaf $\ker(\mathcal{O}_X \to f_*\mathcal{O}_Y)$, and given an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, the support of the sheaf $\mathcal{O}_X/\mathcal{I}$ is a closed subset $Y \subseteq X$ and we can equip it with a structure of a locally ringed space such that the push forward of $\mathcal{O}_Y$ identifies with $\mathcal{O}_X/\mathcal{I}$ (as $\mathcal{O}_X$-algebras).[14]

[13] This can for example be checked by using the adjunction between f^{-1} and Rf_*.

[14] If X is a scheme, then quasi-coherence of $\mathcal{I}$ is equivalent to the fact that the resulting Y is a scheme.

Example 3.7. If $s \in \Gamma(X, \mathcal{O}_X)$ is a section, then we can define its vanishing locus $V(s) := \{x \in X \mid s_x \in \mathfrak{m}_{X,x}\}$ and equip it with the sheaf of rings $\mathcal{O}_X/(s)$. In view of Example 3.6 it corresponds to the ideal $\mathrm{im}(\mathcal{O}_X \xrightarrow{s} \mathcal{O}_X)$. If $f\colon Y \to X$ is a morphism of locally ringed spaces, then $f^{-1}(V(s)) := Y \times_X V(s) \cong V(f^\sharp(s))$ as is easily checked. If $s_1, \ldots, s_n \in \Gamma(X, \mathcal{O}_X)$ then we set $V(s_1, \ldots, s_n) := V(s_1) \times_X \ldots \times_X V(s_n)$. Then $V(s_1, \ldots, s_n) \hookrightarrow X$ is a closed immersion.

Quasi-coherent sheaves can be defined in general, but adding good finiteness properties on $\mathcal{O}_X$-modules requires strong assumptions.

Definition 3.8. Let $(X, \mathcal{O}_X)$ be a locally ringed space, and $\mathcal{M}$ an $\mathcal{O}_X$-module.

1. $\mathcal{M}$ is called quasi-coherent if locally on X there exists a presentation $\mathcal{O}_X^{\oplus I} \to \mathcal{O}_X^{\oplus J} \to \mathcal{M} \to 0$.
2. $\mathcal{M}$ is called finitely generated if locally on X there exists a surjection $\mathcal{O}_X^n \to \mathcal{M}$ for some $n \geq 0$.
3. $\mathcal{M}$ is called finitely presented if locally on X there exists a presentation $\mathcal{O}_X^n \to \mathcal{O}_X^m \to \mathcal{M} \to 0$ for some $n, m \geq 0$.
4. $\mathcal{M}$ is called coherent if $\mathcal{M}$ is finitely generated and for any open subset $U \subseteq X$, any $n \geq 0$ and any morphism $\varphi\colon \mathcal{O}_U^n \to \mathcal{M}$ the kernel $\ker(\varphi)$ is finitely generated.

It is a general fact that the coherent $\mathcal{O}_X$-modules form an abelian subcategory of $\mathrm{Mod}_{\mathcal{O}_X}$, but this subcategory may very well be $\{0\}$ or $\mathcal{O}_X$ need not be coherent.[15]

We will need the following definition for the notion of complex analytic spaces.

Definition 3.9. We call a closed immersion $f\colon Y \to X$ of locally ringed spaces finitely presented if $\ker(f^\sharp\colon \mathcal{O}_X \to f_*\mathcal{O}_Y)$ is (locally) finitely generated.

3.10. **Complex analytic spaces.** We can now give a very clean definition of a complex analytic space. For an open subset $U \subseteq \mathbb{C}^n$ we denote by $\mathcal{O}_U$ its sheaf of holomorphic functions.

Definition 3.11. A complex analytic space is a locally ringed space $(X, \mathcal{O}_X)$ over $\mathrm{Spec}(\mathbb{C})$, which locally admits a finitely presented closed immersion into some $(U, \mathcal{O}_U)$ for some open subset $U \subseteq \mathbb{C}^n$. Morphisms of complex analytic spaces are morphisms of locally ringed spaces over $\mathrm{Spec}(\mathbb{C})$.

Thus, locally $(X, \mathcal{O}_X)$ is isomorphic to the vanishing locus

$$V(f_1, \ldots, f_r) := \{(z_1, \ldots, z_n) \in U \subseteq \mathbb{C}^n \mid f_i(z_1, \ldots, z_n) = 0 \text{ for all } i = 1, \ldots, r\}$$

for *finitely many* holomorphic functions $f_1, \ldots, f_r\colon U \to \mathbb{C}$ on some open subset $U \subseteq \mathbb{C}^n$, where the vanishing locus is given the unique sheaf of rings whose pushforward to U is the sheaf $\mathcal{O}_U/(f_1, \ldots, f_r)$.

Example 3.12. Let X be a complex manifold and let $\mathcal{O}_X$ be its sheaf of holomorphic functions. Then $(X, \mathcal{O}_X)$ is a complex analytic space. In general complex analytic spaces can have singularities and nilpotent elements in their structure sheaf. If $f\colon Y \to X$ is a holomorphic map between two complex manifolds, then f and the pullback $f^\sharp(g) := g \circ f$ define a morphism $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ of locally ringed spaces. We denote this morphism of locally ringed spaces again by f.

We record the following theorem about the local structure of complex analytic spaces.

Theorem 3.13. *Let $(X, \mathcal{O}_X)$ be a complex analytic space.*

1. *(Oka's coherence theorem) The sheaf $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module.*
2. *For any $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a local noetherian ring.*

Proof. Cf. [27, Section 4] or [9, p. I.10]. We note that the second point follows easily by Weierstraß preparation theorem Theorem 3.14. Clearly, the claim reduces to the case that $X = \mathbb{C}^n$ and $x = 0$. Set $R_n = \mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z_1, \ldots, z_n\}$. If $I \subseteq R_n$ is a non-zero ideal, then choose $f \in I$ non-zero. By Weierstraß preparation (and a linear change of coordinates if necessary) we may assume that f is a monic polynomial in z_n and coefficients in R_{n-1}. This implies that $R_n/(f)$ is a finitely generated R_{n-1}-module. By induction we get that R_{n-1} is noetherian and then we can conclude that $I/(f) \subseteq R_n/(f)$ is finitely generated as desired. □

We used the preparation theorem, which builds a backbone for the theory of functions of several complex variables.

[15]Exercise: Let R be a ring and M an R-module. Call M a coherent R-module if M is finitely generated and for any morphism $\varphi\colon R^n \to M$ (not necessarily surjective!) the kernel $\ker(\varphi)$ is finitely generated. Show that the coherent R-modules form an abelian category (or prove the more general assertion on coherent $\mathcal{O}_X$-modules). Assume now that R is a valuation ring. Show that R is a coherent R-module.

Theorem 3.14 (Weierstraß preparation theorem). *Let $f \in \mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z_1,\ldots,z_n\}$ be a convergent power series and assume that f is z_n-general, i.e., $f(0,\ldots,0,z_n) \neq 0$. Then there exists a unique factorization $f = h \cdot w$ with $h \in \mathbb{C}\{z_1,\ldots,z_n\}^\times$ a unit and $w \in \mathbb{C}\{z_1,\ldots,z_{n-1}\}[z_n]$ a Weierstraßpolynomial, i.e., $w(z_1,\ldots,z_n) = z_n^d + g_1(z_1,\ldots,z_{n-1})z_n^{d-1} + \ldots + g_d(z_1,\ldots,z_{n-1})$ and $g_i(0,\ldots,0) = 0$ for $i = 1,\ldots,d$.*

Proof. Cf. [9, p. I.3.3]. □

Geometrically, the Weierstraß preparation theorem implies that the vanishing locus of a z_n-general f is a "branched" cover over $\mathbb{C}^{n-1}$. Next we identify morphisms of complex analytic spaces to $\mathbb{C}^n$.

Proposition 3.15. *Let $(X,\mathcal{O}_X)$ be a complex analytic space and $n \geq 0$. Then evaluating at the coordinate projections $z_i \in \mathcal{O}_{\mathbb{C}}(\mathbb{C}^n)$ defines a bijection*

$$\Phi\colon \mathrm{Hom}_{(\mathrm{lrs}/\mathrm{Spec}(\mathbb{C}))}((X,\mathcal{O}_X),(\mathbb{C}^n,\mathcal{O}_{\mathbb{C}^n})) \to \Gamma(X,\mathcal{O}_X)^n,\ f \mapsto (f^\sharp(z_i))_i.$$

Proof. Assume that $f = (f_0,f^\sharp), g = (g_0,g^\sharp)\colon X \to \mathbb{C}^n$ are two morphisms of locally ringed spaces with $\Phi(f) = \Phi(g)$. Let $\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n$. Then note that

$$\{\lambda\} = V(z_1-\lambda_1,\ldots,z_n-\lambda_n) := V(z_1-\lambda_1)\cap\ldots\cap V(z_n-\lambda_n).$$

As f is a morphism of locally ringed spaces over $\mathbb{C}$, we can conclude that

$$f_0^{-1}(\{\lambda\}) = V(f^\sharp(z_1)-\lambda_1,\ldots f^\sharp(z_n)-\lambda_n),$$

i.e., f_0 is determined by $f^\sharp$. This implies that $f_0 = g_0$. Let $x \in X$ and consider the morphisms

$$f^\sharp, g^\sharp\colon \mathcal{O}_{\mathbb{C},\lambda} \to \mathcal{O}_{X,x}$$

for $f_0(x) = g_0(x) = \lambda = (\lambda_1,\ldots,\lambda_n)$. Now, $f^\sharp, g^\sharp$ are local morphisms of local rings, and hence continuous for the adic topologies on both rings. Now, $\mathbb{C}[(z_1-\lambda_1),\ldots,(z_n-\lambda_n)] \subseteq \mathcal{O}_{\mathbb{C},f_0(x)}$ is a dense subring for the adic topology and $f^\sharp = g^\sharp$ on this subring as $\Phi(f) = \Phi(g)$. Hence $f^\sharp = g^\sharp$ on $\mathcal{O}_{\mathbb{C},\lambda}$. As $x \in X$ was arbitrary this finishes the proof that $f = g$. Now we show surjectivity. First we assume that X is a complex manifold. As $\Gamma(X,\mathcal{O}_X)$ is exactly the ring of holomorphic functions $X \to \mathbb{C}$ and holomorphic functions induces morphisms of locally ringed spaces, we can deduce surjectivity in this case. Now assume that X is a general complex analytic space. As morphisms of complex analytic spaces and sections glue, we are allowed to shrink X. Take a collection of global sections $s_1,\ldots,s_n \in \Gamma(X,\mathcal{O}_X)$. After shrinking X we may assume that there exists a complex manifold V containing X and global sections $f_1,\ldots,f_n \in \Gamma(V,\mathcal{O}_V)$, i.e., holomorphic functions $f_i\colon V \to \mathbb{C}$, which restrict to $s_1,\ldots,s_n$ on X. From the proven case of V, we can now deduce that s is induced by restricting the morphism of locally ringed spaces induced by f. □

Remark 3.16. From Proposition 3.15 we can deduce that the category of complex analytic spaces has all finite limits.[16] More precisely, the claim reduces (via glueing and monomorphisms to some $\mathbb{C}^n$) to the case of products of $\mathbb{C}^n$'s, which reduces to Proposition 3.15, and equalizers. But the case of equalizers reduces to the statement that if $f\colon Y \to X$ is a finitely presented closed immersion of locally ringed spaces and X a complex analytic space, then Y is a complex analytic space (almost by definition).

In contrast to the case of schemes the functor

$$X \mapsto |X|$$

from complex analytic spaces to topological spaces commutes with finite limits.[17] This is quite useful.

We end our discussion of complex analytic spaces by introducing the classes of separated, proper, finite and smooth morphisms.

Let us recall that a continuous map $g\colon T \to S$ of topological spaces is called separated if its diagonal $\Delta_f\colon T \to T\times_S T$ has closed image. Let us mention the following equivalent characterizations of proper maps.

Lemma 3.17. *Let $f\colon T \to S$ be a continuous map of topological spaces. Then the following conditions are equivalent:*

1. *f is universally closed, i.e., for all continuous maps $Z \to S$ the map $T\times_S Z \to Z$ is closed.*
2. *f is closed, and the preimage of each quasi-compact subset $W \subseteq S$ is quasi-compact.*

[16] From the case of products one sees that these don't agree with products (over Spec($\mathbb{C}$)) in the category of locally ringed spaces.

[17] For products this follows from Proposition 3.15 and for equalizer this reduces to the vanishing locus of holomorphic functions, which is easy.

(3) *f is closed and for all $s \in S$ the fiber $f^{-1}(s)$ is quasi-compact.*

If these conditions are satisfied and f is separated, we call f a proper map.

Proof. For simplicity we assume that T, S are second countable to argue via sequences, and that each point in T, S is closed. The general case follows by similar arguments. Assume (1). Then f is closed. Let $W \subseteq S$ be quasi-compact. We have to check that $f^{-1}(W) = W \times_S T$ is quasi-compact. Replacing S by W we may assume that $W = S$, i.e., S quasi-compact. Assume that $t_1, t_2, \ldots \in T$ is a sequence, which does a convergent subsequence. Passing to a subsequence we may assume that $f(t_1), f(t_2), \ldots$ converge to some $s \in S$ as S is quasi-compact. Set $Z := \mathbb{N} \cup \{\infty\}$ as the one-point compactification of S. Then $n \mapsto f(t_n), \infty \mapsto s$ defines a continuous map $Z \to S$. As $t_1, t_2, \ldots$ contains no convergent subsequence the set $A := \{(n, t_n) \in Z \times_S T \mid n \in \mathbb{N}\} \subseteq Z \times_S T$ is actually closed. Let $(z, t) \in Z \times_S T \subseteq A$. We have to show that there exists an open neighborhood of (z, t) not meeting A. If $z \neq \infty$ this is clear as we assumed the t_n to be closed. If $z = \infty$, then there exists an neighborhood $U \subseteq T$ of t such that U does not contain any $t_n, n \in \mathbb{N}$. Then A cannot meet the open neighborhood $Z \times_S U$ of z. By assumption we get that A maps to a closed subset of Z, i.e., onto Z as $\mathbb{N}$ is contained in the image of A. This is a contradiction as a preimage in A of ∞ would yield a convergent subsequence. The implication (2) $\Rightarrow$ (3) is trivial. Assume (3) and let $g \colon Z \to S$ be continuous and $A \subseteq Z \times_S T$ closed. Let $p \colon Z \times_S T \to Z$ and $q \colon Z \times_S T \to T$ be the projections. Assume now that $a_1, a_2, \ldots$ is a sequence of points in A, and $z \in Z$, such that $p(a_i), i \in I$, converges to z, i.e., z is a boundary point of A. It suffices to check that $t_1 := q(a_1), t_2 := q(a_2), \ldots$ has a convergent subsequence with a limit t mapping to $g(z) \in S$, because then the point $(t, z) \in \overline{A} = A$ will map to z. Assume that $t_1, t_2, \ldots$ has no accumulation point in $f^{-1}(g(z))$. Then for each $x \in f^{-1}(g(z))$ there exists an open neighborhood $U_x \subseteq T$, such that U_x only contains t_i only for finitely many i. Set $U := \bigcup\limits_{x \in f^{-1}(g(z))} U_x$, which is an open neighborhood of $f^{-1}(g(z))$. As $f^{-1}(g(z))$ is quasi-compact we may replace U by some open neighborhood of $f^{-1}(g(z))$, which contains t_i only for finitely many i. As f is closed there exists an open neighborhood $V \subseteq S$ of $g(z)$ such that $f^{-1}(V) \subseteq U$. By construction of U the set V can contain $f(t_i)$ only for finitely many i. But this is a contradiction to the convergence of $f(t_1) = g(p(a_1)), \ldots$ to $g(z) \in S$. This finishes the proof. □

Now we pass to complex analytic spaces.

Definition 3.18. Let $f \colon Y \to X$ be a morphism of complex analytic spaces.

(1) The morphism f is called separated if $\Delta_f \colon Y \to Y \times_X Y$ is a closed immersion (or equivalently, by Example 3.4, if $\Delta_f(Y)$ is closed).
(2) The morphism f is called proper if f is separated and $|f| \colon |Y| \to |X|$ is a proper map of topological spaces.

Example 3.19. If $X = \{*\}$ is a point, then $f \colon Y \to X$ is separated if and only if Y is Hausdorff, and it is proper if and only if Y is compact (and in particular, Hausdorff).

Definition 3.20. Let $f \colon Y \to X$ be a morphism of complex analytic spaces.

(1) The morphism f is called quasi-finite at $y \in$ if y is discrete in $f^{-1}(f(y))$. If f is quasi-finite at any $y \in Y$, then f is called quasi-finite.
(2) The morphism f is called finite if f is quasi-finite and proper.

Example 3.21. If $f \colon Y \to X$ is a topological covering, then f is quasi-finite. If f has additionally finite fibers, then f is finite in the sense of Definition 3.20. Note that in general finite fibers are not enough to ensure that a morphism is finite.

Finally, we give the following very intuitive definition of a smooth morphism, cf. [15, Théorème 3.1].

Definition 3.22. Let $f \colon Y \to X$ be a morphism of complex analytic spaces, and $y \in Y$. Set $x := f(y)$. Then f is called smooth at x if for some $n \geq 0$ there exists open neighborhoods U of x and V of y with $f(V) \subseteq U$, an open subset $W \subseteq \mathbb{C}^n$ and an isomorphism $V \cong U \times W$ of locally ringed spaces over U. If f is smooth at any point $y \in Y$, then we call f smooth. Finally, we call f étale if f is smooth and quasi-finite.

The following lemma is clear.

Lemma 3.23. *Let $f \colon Y \to X$ be a morphism of complex analytic spaces. Then f is étale if and only if f is a local isomorphism.*

Proof. If f is a local isomorphisms, then clearly f is smooth and quasi-finite, i.e., étale. Conversely, if f is smooth and quasi-finite, then locally f is isomorphic to the projection $W \times X \to X$ for some open subset $W \subseteq \mathbb{C}^n$. But being quasi-finite forces $n = 0$ and f is a local isomorphism. □

Further properties of complex analytic spaces (or morphisms between them) will be introduced when necessary.

3.24. **Analytification.** In essence analytification associates to the scheme given by the zero locus $V(f_1, \ldots, f_r) \subseteq \mathbb{A}^n_{\mathbb{C}}$ of some polynomials $f_1, \ldots, f_r \in \mathbb{C}[T_1, \ldots, T_n]$ the complex analytic space associated with the zero set in $\mathbb{C}^n$ of the holomorphic functions $f_1, \ldots, f_r \colon \mathbb{C}^n \to \mathbb{C}$ induced by the polynomials $f_1, \ldots, f_r$. A priori, this could depend on the choice of an embedding into $\mathbb{A}^n_{\mathbb{C}}$. To remedy this one introduces analytification by a universal property as follows, cf. [14, Exposé XII].

Let X be a scheme over $\mathbb{C}$, which is locally of finite type.

Theorem 3.25 ([14, Exposé XII.Théorème et Définition 1.1])**.** *There exists a complex analytic space X^{an} with a morphism $\varphi \colon X^{\mathrm{an}} \to X$ of locally ringed spaces (over* $\mathrm{Spec}(\mathbb{C})$*) such that for any complex analytic space Z the morphism φ induces a bijection*

$$\mathrm{Hom}_{(\mathrm{lrs}/\mathbb{C})}(Z, X^{\mathrm{an}}) \to \mathrm{Hom}_{(\mathrm{lrs}/\mathbb{C})}(Z, X).$$

Clearly, the complex-analytic space X^{an} (with φ) is unique up to unique isomorphism. Moreover, for a morphism $f \colon Y \to X$ of schemes, locally of finite type over $\mathbb{C}$, we get a morphism $f^{\mathrm{an}} \colon Y^{\mathrm{an}} \to X^{\mathrm{an}}$.

Proof. We first note the following permance properties, which follow from the universal property of analytification and properties of the category of locally ringed spaces:

(1) If $Y \subseteq X$ is an open subscheme and X^{an} exists, then Y^{an} exists and $Y^{\mathrm{an}} \cong X^{\mathrm{an}} \times_X Y = \varphi^{-1}(Y)$.
(2) If $Y \subseteq X$ a closed subscheme defined by a quasi-coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$ and X^{an} exists, then Y^{an} exists and $Y^{\mathrm{an}} \cong X^{\mathrm{an}} \times_X Y$ is the vanishing locus of the ideal $\mathcal{I} \cdot \mathcal{O}_{X^{\mathrm{an}}}$ (i.e., the universal locally ringed space over X^{an} such that $\mathcal{I} \cdot \mathcal{O}_{X^{\mathrm{an}}}$ is send to 0).
(3) If Y is another scheme locally finite type over $\mathbb{C}$ and $Y^{\mathrm{an}}, X^{\mathrm{an}}$ exist, then $(X \times_{\mathrm{Spec}(\mathbb{C})} Y)^{\mathrm{an}}$ exists and in fact is isomorphic to $X^{\mathrm{an}} \times_{\mathrm{Spec}(\mathbb{C})} Y^{\mathrm{an}}$. More generally, analytification commutes with finite limits.
(4) If $X = \bigcup\limits_{i \in I} U_i$ is an open cover and U_i^{an} exists for all $i \in I$, then X^{an} exists and $X^{\mathrm{an}} = \bigcup\limits_{i \in I} U_i^{\mathrm{an}}$.

For the second point one needs to use that $\mathcal{I}$ is locally generated by finitely many global sections, and for the forth point one uses that complex analytic spaces can be constructed by glueing and that the universal property (plus (1)) allow to guarantee the cocycle condition on the overlaps $U_i^{\mathrm{an}} \cap U_j^{\mathrm{an}}$.

These permance properties imply that it suffices to construct the analytification of $\mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[z])$. But in this case Proposition 3.15 implies that the analytification is simply $\mathbb{C}$ with its usual sheaf of holomorphic functions (and φ is induced by the inclusion $\mathbb{C}[z] \subseteq \mathcal{O}_{\mathbb{C}},\ z \mapsto (\mathbb{C} \to \mathbb{C},\ y \mapsto y)$). □

We record the following properties of X^{an} and $\varphi \colon X^{\mathrm{an}} \to X$.

Theorem 3.26. (1) *The map φ induces a bijection $|X^{\mathrm{an}}| \to X(\mathbb{C})$.*
(2) *For any $x \in X^{\mathrm{an}}$ the map $\mathcal{O}_{X,\varphi(x)} \to \mathcal{O}_{X^{\mathrm{an}},x}$ induces an isomorphism*

$$\mathcal{O}^{\wedge}_{X,\varphi(x)} \cong \mathcal{O}^{\wedge}_{X^{\mathrm{an}},x}$$

on completions. In particular, the map $\mathcal{O}_{X,\varphi(x)} \to \mathcal{O}_{X^{\mathrm{an}},x}$ is (faithfully) flat for each $x \in X^{\mathrm{an}}$.
(3) *The pullback functor $\varphi^* \colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_{X^{\mathrm{an}}}}$ is exact, faithful and conservative. In particular, φ is a flat morphism of locally ringed spaces.*

Proof. The first point follows from the universal property of X^{an}, cf. Theorem 3.25, applied with $Z = \mathrm{Spec}(\mathbb{C})$. The second point maybe proven locally on X, and hence we may assume that $X = V(f_1, \ldots, f_r) \subseteq \mathbb{A}^n_{\mathbb{C}}$ is the vanishing locus of some polynomials. By construction we get $X^{\mathrm{an}} = V(f_1, \ldots, f_r) \subseteq \mathbb{C}^n$ and from the definition of the structure sheaf on this vanishing locus we see that the claim reduces to $X = \mathbb{A}^n_{\mathbb{C}}$, in which case both completed local rings are explicit power series rings (and φ obviously an isomorphism). The claim on faithfully flatness follows from this as both local rings $\mathcal{O}_{X,\varphi(x)}, \mathcal{O}_{X^{\mathrm{an}},x}$ are noetherian (and hence their completions are faithfully flat). The third point follows from the second as all properties can be checked on stalks (recall that a morphism of $\mathcal{O}_X$-modules is zero if it is zero in each stalk for all *closed* points of X). □

Example 3.27. It is easy to see, e.g., by comparing the transition maps of the standard cover, that the analytification of $\mathbb{P}^n_{\mathbb{C}}$ is $\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$. From the universal property of analytification we can therefore deduce that morphisms to $\mathbb{C}P^n$ from a complex analytic space X identify with isomorphism classes of pairs $(\mathcal{L}, \alpha)$ with $\mathcal{L}$ a line bundle on X and $\alpha\colon \mathcal{O}^{n+1} \to \mathcal{L}$ a surjection.

3.28. **Permance of properties under analytification, part I.** Let X be a scheme over $\mathbb{C}$, which is locally of finite type, and let X^{an} be its analytification. We analyze now which properties of X are reflected on X^{an} and vice versa.

Lemma 3.29 ([14, Exposé XII.Proposition 2.1])**.** *Let P be one of the following properties:*

(1) *non-empty,*
(2) *discrete,*
(3) *regular,*
(4) *normal,*
(5) *reduced,*
(6) *of dimension n (for some fixed $n \geq 0$).*

Then X satisfies P if and only if X^{an} satisfies P.

Proof. Being non-empty is equivalent to $X(\mathbb{C}) \neq \emptyset$ as X is locally of finite type over $\mathbb{C}$. Hence, (1) follows. Clearly, X discrete implies X^{an} discrete as analytification preserves disjoint unions. Conversely, assume that X^{an} is discrete. By Noether normalization for analytic algebras, cf. [10, Lemma 1.12] or [9, p. 7.2], this implies that $\dim(\mathcal{O}_{X^{\mathrm{an}},x}) = 0$ for each $x \in X$, i.e., that $\mathcal{O}_{X^{\mathrm{an}},x}$ is finite dimensional over $\mathbb{C}$ for any $x \in X(\mathbb{C})$. As then $\mathcal{O}_{X,x} \hookrightarrow \mathcal{O}^\wedge_{X,x} \cong \mathcal{O}^\wedge_{X^{\mathrm{an}},x} \cong \mathcal{O}_{X^{\mathrm{an}},x}$, we get that $\mathcal{O}_{X,x}$ is finite dimensional over $\mathbb{C}$. This implies that $x \in X$ is an open point as desired. This finishes the proof of (2). For any local noetherian ring R we know that R is regular if and only if $R^\wedge$ is regular, cf. [Stacks, Tag 07NU]. Similarly, $\dim(R) = \dim(R^\wedge)$. This handles (3) and (6). The remaining two cases are more difficult and use the *excellence* of the local noetherian rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X^{\mathrm{an}},x}$, cf. [Stacks, Tag 07QW]. Namely, excellence of a local noetherian ring R implies that R is normal (reduced,...) if and only if $R^\wedge$ is normal (reduced, ...), cf. [Stacks, Tag 07QS]. □

Let us make the following definition.

Definition 3.30. An analytic algebra A is a quotient of $\mathbb{C}\{z_1, \ldots, z_n\}$. Note that analytic algebras are local noetherian rings with residue field $\mathbb{C}$ (and exactly the local rings of complex analytic spaces).

In Lemma 3.29 we used a special property of these rings, namely excellence. Let us discuss this property a bit.

3.31. **Digression on excellent rings.** Excellence of local noetherian rings implies a good interaction with completions.

Definition 3.32. Let A be a local noetherian ring and $A^\wedge$ is completion. The formal fibers of A are the fibers of the morphism $\mathrm{Spec}(A^\wedge) \to \mathrm{Spec}(A)$.

The formal fibers are always noetherian schemes, but they can behave arbitrary bad in principle.

Example 3.33 ([Stacks, Tag 02JD])**.** We present an example where the formal fibers are non-reduced. Set $K := \mathbb{C}\{x\}[1/x]$ as the field of convergent Laurent series. Algebraically, K has infinite transcendence degree over $\mathbb{C}$. Hence, we may choose $f_n \in x\mathbb{C}\{x\}, n \geq 1$ such that in $\Omega^1_{K/\mathbb{C}}$ the derivatives $dx, df_1, \ldots$ are linearly independent over K. As

$$\mathrm{Hom}_K(\Omega^1_{K/\mathbb{C}}, \mathbb{C}((x))) = \mathrm{Der}_{\mathbb{C}}(K, \mathbb{C}((x)))$$

we can conclude that there exists a derivation $D\colon \mathbb{C}\{x\} \to \mathbb{C}((x))$ such that $D(x) = 0$ and $D(f_n) = x^{-n}$ for $n \geq 1$ (we may for simplicity assume that D sends complementary basis vectors in $\Omega^1_{K/\mathbb{C}}$ to 0). Set

$$A := \{f \in \mathbb{C}\{x\} \mid D(f) \in \mathbb{C}[[x]]\}.$$

Now the following statements hold:

1. $A \hookrightarrow \mathbb{C}\{x\}$ is an integral, birational extension, in particular A is a local domain of dimension 1,
2. the maximal ideal of A is given by (x, xf_1), in particular A is a noetherian ring[18],
3. the maps $\mathbb{C}[x]_{(x)} \to A \to \mathbb{C}\{x\}$ extend to maps $\mathbb{C}[[x]] \xrightarrow{\iota} A^\wedge \xrightarrow{\psi} \mathbb{C}[[x]]$ such that $\psi \circ \iota = \mathrm{Id}$,
4. the derivation $D\colon A \to \mathbb{C}[[x]]$ extends to a continuous derivation $\hat{D}\colon A^\wedge \to \mathbb{C}[[x]]$,
5. $\hat{D}$ is zero on $\iota(\mathbb{C}[[x]])$,
6. there exists a ring isomorphism $A^\wedge \cong \mathbb{C}[[x]][\varepsilon]$, $a \mapsto \psi(a) + \hat{D}(a)\varepsilon$, with $\varepsilon^2 = 0$.

Let $0 \neq f \in \mathbb{C}\{x\}$. We may assume that $f \in x\mathbb{C}\{x\}$. Then $D(f) = h/f^n$ for some $n \geq 0$ and $h \in \mathbb{C}[[x]]$. Hence, $D(f^{n+1}) = (n+1)f^n D(f), D(f^{n+2}) = (n+2)f^{n+1}D(f) \in \mathbb{C}[[x]]$, i.e., $f^{n+1}, f^{n+2} \in A$. This proves (1) as the fraction field of A must be $K = \mathrm{Frac}(\mathbb{C}\{x\})$. Now $\mathrm{Spec}(\mathbb{C}\{x\}) \to \mathrm{Spec}(A)$ must be surjective and this implies that $A \cap x\mathbb{C}\{x\}$ is the unique maximal ideal $\mathfrak{m}_A$ of A. Let $f \in \mathfrak{m}_A$. Then $h := D(f) \in \mathbb{C}[[x]]$. Write $h = c + xh'$ with $h' \in \mathbb{C}[[x]]$. Then $D(f - cxf_1) = c + xh' - cxD(f_1) = c + xh' - c = xh'$. But $f - cxf_1 = xg$ with $g \in \mathbb{C}\{x\}$. Now, $xD(g) = xh'$, which implies $D(g) = h' \in \mathbb{C}[[x]]$, i.e., $g \in A$. Therefore, $f = cxf_1 + xg \in (x, xf_1)$ as desired. The existence of the maps ι, ψ is clear. Let us check that D is $\mathfrak{m}_A$-adically continuous. But $D(\mathfrak{m}_A^n) \subseteq (x)^{n-1}$ by an easy induction.

Let us check that $A^\wedge$ is non-reduced (leaving the verifiation that the map is an isomorphism to the literature). As $A^\wedge$ is 1-dimensional and admits $\mathrm{Spec}(\mathbb{C}[[x]])$ as a closed subscheme via ψ, it suffices to see that $\hat{D}\colon A^\wedge \to \mathbb{C}[[x]][\varepsilon]$ hits ε. As $D(x) = 0$ we see that $\hat{D}(\iota(h)) = 0$ for any $h \in \mathbb{C}[[x]]$. Now consider the element $a_n := x^n f_n \in A \subseteq A^\wedge$. Then $\hat{D}(a_n) = 1$, which implies that $\psi(a_n) = x^n f_n + \varepsilon$. As the element $x^n f_n \in \mathbb{C}\{x\}$ lies in the image of ι, we see that $\varepsilon \in \mathbb{C}[[x]]$ is in the image of $A^\wedge$. This implies the desired non-reducedness of $A^\wedge$. Explicitly, if we set $h := x^n f_n \in \mathbb{C}[[x]]$, then $a_n - \iota(h)$ is a preimage of ε (beware that $\iota(h) \neq a_n$!).

Let us give now the definition of an excellent local ring.

Definition 3.34. A noetherian ring A is called excellent if the following conditions are satisfied:

1. For any prime $\mathfrak{p}$ of A the formal fibers of $A_{\mathfrak{p}}$ are geometrically regular.
2. For any finite A-algebra B, the regular locus of $\mathrm{Spec}(B)$ is open.
3. For each A-algebra B of finite type and any two primes $\mathfrak{p} \subseteq \mathfrak{q}$ in B any two saturated chains of primes between $\mathfrak{p}$ and $\mathfrak{q}$ have the same length.

Luckily there are many example of excellent rings.

Theorem 3.35. *The following rings are excellent:*

1. *fields,*
2. *complete local noetherian rings,*
3. $\mathbb{Z}$*,*
4. *Dedekind domains with fraction field of characteristic* 0*,*
5. *finite type extensions of any excellent rings, and localizations of excellent rings.*
6. *rings of convergent power series over* $\mathbb{R}$ *or* $\mathbb{C}$*.*

[18] A commutative ring is noetherian if and only if all its prime ideals are noetheran, cf. [11, Chapitre 0, Proposition (6.4.7.)].

Proof. Cf. [Stacks, Tag 07QW] and [20, Chapter 13]. □

Theorem 3.36. *Let $\varphi\colon A \to B$ be a flat morphism of noetherian rings.*

(1) *If the fibers of* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *are geometrically reduced, then B is reduced if A is reduced.*
(2) *If the fibers of* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *are geometrically normal, then B is normal if A is normal.*

If φ is faithfully flat, then the "if" can be replaced by an "if and only if".

Proof. This is [Stacks, Tag 0C21], [Stacks, Tag 0C22]. The additional statements if φ is faithfully flat are easier, cf. [Stacks, Tag 033F], [Stacks, Tag 033G]. □

In particular, for an excellent local noetherian ring A we can deduce that A is normal (reduced,...) if and only if $A^\wedge$ is normal (reduced,...). Another interesting case is when B is étale over A. Then we can conclude that A being normal (reduced) etc. is "étale-local". Note that using approximation with noetherian rings we can deduce Theorem 3.36 for any étale map $A \to B$ of rings.

Here, we end our digression on excellent rings.

3.37. **Permanence of properties under analytification, part II.** Let X be a scheme over $\mathbb{C}$, which is locally of finite type. We now analyze permanence of topological properties under analytification.

Lemma 3.38 ([14, Exposé XII. Proposition 2.3])**.** *Let $Z \subseteq X$ be a locally constructible subset. Then Z is closed (resp. open) if and only if $\varphi^{-1}(Z)$ is closed (resp. open).*

The condition that Z is a locally constructible subset means that locally Z is a finite union of locally closed subsets, cf. [Stacks, Tag 005L], and we can take the latter as a definition for now.

Proof. The "only if" part is clear by the proof of Theorem 3.25. Note that the claim is local and compatible with finite unions in Z. Hence, we may assume that Z is locally closed. By passing to the complement it suffices to deal with the case that $\varphi^{-1}(Z)$ is closed. Replacing X by $\overline{Z}$ we may reduce to the case that $Z \subseteq X$ is open and dense. We may also assume that $X = \mathrm{Spec}(R)$ is a reduced affine scheme. Assume that $X \setminus Z$ is a non-empty closed subscheme, defined by some ideal $I \subseteq \mathcal{O}_X(X)$. We know that $\varphi^{-1}(V(I)) \subseteq X^{\mathrm{an}}$ is open (as it is the complement of $\varphi^{-1}(Z)$). This implies that for each $\mathbb{C}$-rational point $x \in V(I) \subseteq X$ the image of I in the stalk $\mathcal{O}_{X^{\mathrm{an}},x}$ vanishes (this uses Rückert's Nullstellensatz, cf. [10, Statement (8.3)′]). As $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathrm{an}},x}$ is injective (being faithfully flat), this implies that $V(I)$ contains an open neighborhood of x in X. As Z is dense this implies that $Z \cap V(I) \neq \emptyset$, which is a contradiction. Hence, $Z = X$ is closed. □

The following statement is surprisingly difficult.

Proposition 3.39. *X is connected if and only if X^{an} is connected.*

Proof. If X^{an} is connected, then X is connected as $X^{\mathrm{an}} \to X$ is continuous with imag $X(\mathbb{C})$ (and X connected is equivalent to its subspace $X(\mathbb{C})$ being connected). Conversely, assume that X is connected. Then each irrducible component of X meets another one, and thus it suffices to check the statement if X is irreducible. As the normalizaton $\tilde{X} \to X$ is surjective, also $\tilde{X}^{\mathrm{an}} \to X^{\mathrm{an}}$ is surjective. Hence, we may assume that X is irreducible and normal. As non-empty opens in an irreducible space have non-trivial intersection, we may furthermore reduce to the case that X is affine. Then we can find a normal, connected projective $\mathbb{C}$-scheme P and an open immersion $X \to P$. Using GAGA we will later check the statement for P, cf. Remark 3.53. By Theorem 3.40 the connected components of X and P don't change if we remove the singular points or anything of codimension ≥ 2. This reduces us to checking that if P is a connected smooth scheme over $\mathbb{C}$ (not necessarily projective) with P^{an} connected and $D \subseteq P$ is a Cartier divisor with D smooth over $\mathbb{C}$, then $P^{\mathrm{an}} \setminus D^{\mathrm{an}}$ is connected. But by the implicit function theorem, $P^{\mathrm{an}} \setminus D^{\mathrm{an}}$ identifies locally with the vanishing locus of a coordinate in $\mathbb{C}^n$. More precisely, let $j\colon P^{\mathrm{an}} \setminus D^{\mathrm{an}} \to P^{\mathrm{an}}$ be the inclusion. By the local calculation just made we can conclude that the natural map $\underline{\mathbb{Z}} \to j_*(\underline{\mathbb{Z}})$ is an isomorphism by checking this on each stalk (here $\underline{\mathbb{Z}}$ denotes the constant sheaf associated with $\mathbb{Z}$). Now, $\Gamma(P^{\mathrm{an}}, \underline{\mathbb{Z}}) = \mathbb{Z}$ as P^{an} is connected, and thus

$$\Gamma(P^{\mathrm{an}} \setminus D^{\mathrm{an}}, \underline{\mathbb{Z}}) = \Gamma(P^{\mathrm{an}}, j_*(\underline{\mathbb{Z}})) = \mathbb{Z},$$

which implies that $P^{\mathrm{an}} \setminus D^{\mathrm{an}}$ is connected as desired. This finishes the proof. □

Theorem 3.40 (Riemann's extension theorem for normal complex analytic spaces)**.** *Let X be a normal complex space and $A \subseteq X$ a closed analytic subset of codimension ≥ 2. Then $\mathcal{O}(X) \to \mathcal{O}(X \setminus A)$ is bijective. Moreover, the singular locus of X is a closed analytic subset of codimension ≥ 2.*

Proof. Cf. [10, p. 13.6] and [10, (13.2)]. □

Next let us move to properties of morphisms. Let $f\colon Y \to X$ be a morphism of schemes which are locally of finite type over $\mathbb{C}$, and let $f^{\mathrm{an}}\colon Y^{\mathrm{an}} \to X^{\mathrm{an}}$ be its analytification.

Proposition 3.41. *Let P one of the following properties for a morphism:*

(1) *flat,*
(2) *quasi-finite,*
(3) *injective,*
(4) *separated,*
(5) *isomorphism,*
(6) *open immersion,*
(7) *monomorphism,*
(8) *unramified,*
(9) *étale,*
(10) *smooth.*

Then f has P if and only if f^{an} has P.

Here, we define a morphism of complex analytic spaces to be unramified if it is locally on source and target a (finitely presented) closed immersion.

Proof. The claim on flatness follows directly from Theorem 3.26 as it can be checked on completed stalks. Point (2) follows from Lemma 3.29 as P reduces to the fibers over points being discrete (over closed points in the scheme case, cf. Theorem 10.22). The claim on injectivity follows. The claim on separatedness follows from Lemma 3.38 (as the diagonal always has locally closed image). We now want to reduce all statements to the case of étale morphisms. An isomorphism is an open immersion inducing a bijection on $\mathbb{C}$-points, an open immersion is an injective étale morphism, a monomorphism is a morphism whose diagonal is an isomorphisms. If f is smooth, then f^{an} is étale over an affine space, and if case 9) is settled, then f^{an} is smooth. If conversely, f^{an} is smooth, then f is flat. As all fibers of f^{an} are complex manifolds (as f^{an} is smooth), they are regular. But then each fiber of f over a closed point is (geometrically) regular, which implies that f is smooth. This reduces all claims to the case of unramified and étale morphisms. Let $g\colon Z \to W$ be a morphism of complex analytic spaces. If g is unramified, then it is locally on Z a monomorphism and hence its diagonal is an immersion and a local isomorphism, i.e., an open immersion (= universally injective étale morphism). This reduces the statement "f^{an} unramified implies f unramified" to the case of étale morphisms. Now assume that $\Delta_g\colon Z \to Z \times_W Z$ is an open immersion. For example, $g = f^{\mathrm{an}}$ and f unramified (this implies that Δ_f is an open immersion, and by the proof of Theorem 3.25 open immersions analytify to open immersions). This implies that for any point $z \in Z$ with image $w \in W$ the morphism

$$\widehat{g}_z\colon \mathcal{O}^{\wedge}_{W,w} \to \mathcal{O}^{\wedge}_{Z,z}$$

is surjective. Indeed, by construction of finite limits of complex analytic spaces we see that $\mathcal{O}^{\wedge}_{Z\times_W Z,(z,z)} \cong \mathcal{O}^{\wedge}_{Z,z}\widehat{\otimes}_{\mathcal{O}_{W,w}}\mathcal{O}^{\wedge}_{Z,z} \cong \mathcal{O}^{\wedge}_{Z,z}$ via multiplication, where the tensor product is completed (for the adic topologies). Looking on $\mathfrak{m}/\mathfrak{m}^2$ then implies the claim as a morphism of complete local rings with same residue field is surjective if and only if it is on $\mathfrak{m}/\mathfrak{m}^2$. By Proposition 3.45 the surjectivity of $\widehat{g}_z$ implies the surjectivity of $g_z\colon \mathcal{O}_{W,w} \to \mathcal{O}_{Z,z}$. In other words, we can conclude that g is unramified. Thus we have reduced the proof to the case of étale morphisms, which we will prove in Proposition 3.45. □

Let us give the following concrete example, illuminating that f^{an} is étale iff f is étale in the case that Y, X are smooth $\mathbb{C}$-schemes.

Example 3.42. Assume $f\colon Y = \mathbb{G}_m \to X = \mathbb{G}_m,\ t \mapsto t^n$, i.e., f is induced by the map $\mathbb{C}[t^{\pm}] \to \mathbb{C}[t^{\pm}],\ t \mapsto t^n$ (which does not induce an isomorphism $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ for any $y \in Y(\mathbb{C})$). It is clear that

$$f^{\mathrm{an}}\colon Y^{\mathrm{an}} = \mathbb{C}^{\times} \to X^{\mathrm{an}} = \mathbb{C}^{\times},\ z \to z^n.$$

By considering the derivative of f^{an} (or equivalently f) we see that f^{an} is a local isomorphism by the implicit function theorem as its derivative$n \cdot z^{n-1}$ is non-zero at any point in $z \in \mathbb{C}^{\times}$. More generally, we can easily proof that if $f\colon Y \to X$ is a morphism between two *smooth* schemes over $\mathrm{Spec}(\mathbb{C})$, then f is étale if and only if f^{an} is étale. Indeed, from the Jacobian criterion and the implicit function theorem it follows easily that Y^{an} is a complex manifold if Y is smooth. But a morphism between two smooth schemes is étale if and only if it induces an isomorphism on differentials, and this latter means exactly checking that $f^{\mathrm{an}}\colon Y^{\mathrm{an}} \to X^{\mathrm{an}}$ is a local isomorphism (again by the implicit function theorem).

To prove the remaining statements we introduce the following convenient terminology.

Definition 3.43. If A is an analytic algebra, then a module M is called quasi-finite if $\dim_{\mathbb{C}} M/\mathfrak{m}_A M < \infty$.

If $A \to B$ is a $\mathbb{C}$-algebra morphism of analytic algebras, then $A \to B$ is necessarily local. Now if $f\colon A \to B$ is a morphism of analytic algebras and $\widehat{f}\colon A^\wedge \to B^\wedge$ is surjective, then B is a quasi-finite A-module. Indeed, in this case we even have $B/\mathfrak{m}_A B \cong \mathbb{C}$ as $\widehat{f}$ surjective implies that $\mathbb{C} \cong A^\wedge/\mathfrak{m}_A A^\wedge \to B^\wedge/\mathfrak{m}_A B^\wedge$ is surjective, which implies that $\mathfrak{m}_A B^\wedge = \mathfrak{m}_B B^\wedge$. As $B \to B^\wedge$ is faithfully flat this implies that $\mathfrak{m}_A B = \mathfrak{m}_B$.

We note that (for arbitrary $A \to B$) if $A = \mathbb{C}\{z_1, \ldots, z_n\}/I$ and $B = \mathbb{C}\{w_1, \ldots, w_m\}/J$, then $A \to B$ extends to a morphism $\mathbb{C}\{z_1, \ldots, z_n\} \to \mathbb{C}\{w_1, \ldots, w_m\}$ by lifting the images of the residue classes of the z_i.[19]

Lemma 3.44. *Let $\varphi\colon A \to B$ be a $\mathbb{C}$-linear morphism of analytic algebras and M a finite B-module, which is quasi-finite as an A-module. Then M is a finite A-module.*

Proof. By writing $A = \mathbb{C}\{z_1, \ldots, z_n\}/I, B = \mathbb{C}\{w_1, \ldots, w_m\}/J$ and extending φ as explained before, we may assume that $A = \mathbb{C}\{z_1, \ldots, z_n\}$ and $B = \mathbb{C}\{w_1, \ldots, w_m\}$. Adding variables we may even assume that $B = A\{w_1, \ldots, w_m\}$. Now we argue via induction on $m \geq 1$. The induction step $m \mapsto m+1$ is trivial (as $m \geq 1$). Assume $m = 1$ and write $w := w_1$. By assumption $\dim_{\mathbb{C}} M/\mathfrak{m}_A M < \infty$. Let $n_1, \ldots, n_q \in M$ be generators as a B-module, and let $m_1, \ldots, m_r \in M$ map to generators of the $\mathbb{C}$-vector space $M/\mathfrak{m}_A M$. Thus,

$$M = \mathbb{C}m_1 + \ldots + \mathbb{C}m_r + \mathfrak{m}_A B n_1 + \ldots + \mathfrak{m}_A B n_q$$

and we see that M is finite over the subring $R := \mathbb{C} + \mathfrak{m}_A B \subseteq B$. This implies that there exists a monic polynomial $f(T) \in R[T]$, such that $f(w)M = 0$ (e.g. the characteristic polynomial of a matrix expressing the multiplication by $w \in B$ on M). Write $f(T) = T^d + c_1 T^{d-1} + \ldots + c_d$ with $c_1, \ldots, c_d \in R$. By definition of R we see that $c_j(0, \ldots, 0, w) \in \mathbb{C}$ for all $j = 1, \ldots, d$. In particular, $f(w)$ is w-general in the sense of Theorem 3.14. By Weierstraß preparation, Theorem 3.14, this implies that $B/f(w)$ is a *finitely generated free* A-module. As $g(w)M = 0$ the surjection $B^q \to M$, $e_i \mapsto n_i$ of B-modules must factor over $B^q/f(w)$ and the latter is a finitely generated A-module. This proves that M is a finitely generated A-module as desired. ☐

Now, recall that a morphism $f\colon Y \to X$ between schemes, locally finite type over $\mathbb{C}$, is étale if and only if for all $y \in Y$ the map $\mathcal{O}^\wedge_{X,f(y)} \to \mathcal{O}^\wedge_{Y,y}$ is an isomorphisms, and clearly if f^{an} is étale, then $\mathcal{O}^\wedge_{X^{\mathrm{an}},f(y)} \to \mathcal{O}^\wedge_{Y^{\mathrm{an}},y}$ is an isomorphism for any $y \in Y^{\mathrm{an}}$. The critical statement that we are left to prove is thus the following.

Proposition 3.45. *Let $f\colon A \to B$ be a morphism of analytic algebras. If $\widehat{f}\colon A^\wedge \to B^\wedge$ is surjective (resp. injective, resp. bijective), then f has the same property.*

Proof. The claim on injectivity is trivial (by faithfully flatness of completions for local noetherian rings). Assume that $\widehat{f}$ is surjective or bijective. Then B is a finite, local A-algebra by Lemma 3.44 and the discussion after Definition 3.43. In particular, the ideal $\mathfrak{m}_B$ is nilpotent in $B/\mathfrak{m}_A B$ as the latter is an artinian $\mathbb{C}$-algebra. This implies that the $\mathfrak{m}_B$ and the $\mathfrak{m}_A$-adic topologies on B agree. This implies (together with properties of completions for finitely generated modules over local noetherian rings, cf. [Stacks, Tag 00MA]) that the natural morphism $A^\wedge \otimes_A B \to B^\wedge$ is an isomorphism. Thus, by faithfully flat descent along $A \to A^\wedge$ we can conclude. ☐

Here is a sample application of Proposition 3.45

Exercise 3.46. Let X be a complex analytic space. Then X is regular (in the sense of Definition 3.2) if and only if X is a complex manifold.

After having finished the proof of Proposition 3.45 we continue our discussion of permanence of properties under analytification.

Proposition 3.47. *Let $f\colon Y \to X$ be a morphism of finite type (in particular quasi-compact!) between schemes over $\mathbb{C}$, which are locally of finite type. Let $f^{\mathrm{an}}\colon Y^{\mathrm{an}} \to X^{\mathrm{an}}$ be its analytification. Let P be one of the following properties:*

(1) *surjective,*
(2) *closed immersion,*
(3) *proper,*

[19] The convergence condition is easily checked as these lifts vanish at $(w_1, \ldots, w_m) = 0$ because $A \to B$ is a local homomorphism.

(4) *finite.*

Then f has P if and only if f^{an} has P.

Proof. Let $\varphi\colon X^{\mathrm{an}} \to X$ and $\psi\colon Y^{\mathrm{an}} \to Y$ be the canonical morphisms (of locally ringed spaces). It is clear that f surjective implies that f^{an} is surjective (because analytification commutes with fibers and (1) in Lemma 3.29). Conversely, assume that f^{an} is surjective. Then $Y(\mathbb{C}) \to X(\mathbb{C})$ is surjective. As f is quasi-compact this implies that f is surjective as its image is locally constructible, cf. [Stacks, Tag 054K]. Clearly, if f is a closed immersion, then f^{an} is a closed immersion. For schemes a closed immersion is a proper monomorphism, which reduces (2) to (3). Finite morphisms are exactly the proper and quasi-finite morphisms. This reduces (4) to (3). Assume that f is proper. We may localize on X and assume that X is affine. In this case we may find by Chow's lemma some projective, surjective morphism $Z \to Y$ such that $Z \to X$ is projective. Clearly, the analytification of a projective morphism is a proper morphism as $\mathbb{C}P^n$ is compact and closed immersions analytify to closed immersions. This implies that f^{an} is proper. Now assume that f^{an} is proper, i.e., separated and universally closed. By Proposition 3.41 f is separated. In order to see that f is universally closed it suffices to check that for each scheme Z *of finite type* over X the map $Y \times_X Z \to Z$ is closed. Thus, by stability of properness for maps of topological spaces it suffices to see that f is closed. If $Z \subseteq Y$ is closed, then $f(Z) \subseteq X$ is locally constructible, and hence by Lemma 3.38 $f(Z)$ is closed if and only if $\varphi^{-1}(f(Z)) = f^{\mathrm{an}}(\psi^{-1}(Z))$ is closed. But the latter statement is implied by properness of f^{an}. ☐

Exercise 3.48. The assumption that f is quasi-compact in Proposition 3.47 is important. For each of the properties give an example showing that the conclusion fails if quasi-compactness of f is dropped. *Hint: Consider an infinite set of points in $\mathbb{C}$.*

Let us discuss the analytification of abelian varieties as a concrete example.

Lemma 3.49. *Let $X \to \mathrm{Spec}(\mathbb{C})$ be a proper, connected, smooth group scheme, i.e, an abelian variety. Then $X^{\mathrm{an}} \cong V/\Lambda$ is a complex torus, i.e., a quotient of a finite dimensional $\mathbb{C}$-vector space V by some $\mathbb{Z}$-lattice Λ.*

Proof. By Proposition 3.47, Proposition 3.41 and Proposition 3.39 we know that X^{an} is a connected, compact complex manifold. As analytification commutes with products X^{an} is naturally a complex Lie group and as such it comes equipped with its exponential mapping $\exp\colon V := T_0X^{\mathrm{an}} \to X^{\mathrm{an}}$, where T_1X^{an} denotes the tangent space at the identity $0 \in X^{\mathrm{an}}$. It is classical that each abelian variety is commutative, cf. [21, II.4.Corollary 2]. But this implies that exp is a group homomorphism, and it is surjective as it is a local isomorphism and X^{an} connected. As exp is moreover a local isomorphism (it induces an isomorphism on the tangent spaces of the units), its kernel Λ is a discrete subgroup in V. As $V/\Lambda \cong X^{\mathrm{an}}$ is compact, Λ is a $\mathbb{Z}$-lattice. ☐

3.50. **GAGA.** The next topic will be Serre's GAGA theorem on comparing coherent algebraic and analytic sheaves. We will be rather sketchy here.

Let X be a scheme locally of finite type over $\mathbb{C}$. Then the functor

$$(-)^{\mathrm{an}} := \varphi^*\colon \mathrm{Coh}_X \to \mathrm{Coh}_{X^{\mathrm{an}}},\ \mathcal{F} \to \mathcal{F}^{\mathrm{an}} := \varphi^*\mathcal{F}$$

compares algebraic coherent sheaves and analytic coherent sheaves. Under properness assumptions this relation is as nice as possible.

Theorem 3.51 (GAGA)**.** *Let $f\colon Y \to X$ be a morphism of schemes, locally of finite type over $\mathbb{C}$.*

(1) *If f is proper and $\mathcal{F} \in \mathrm{Coh}_X$, then for any $i \geq 0$ the natural map $(R^if_*(\mathcal{F}))^{\mathrm{an}} \to R^if_*^{\mathrm{an}}(\mathcal{F}^{\mathrm{an}})$ is an isomorphism.*

(2) *If $X \to \mathrm{Spec}(\mathbb{C})$ is proper, then the functor $(-)^{\mathrm{an}}\colon \mathrm{Coh}_X \to \mathrm{Coh}_{X^{\mathrm{an}}}$ is an equivalence, i.e., each analytic coherent sheaf is algebraic.*

Sketch of proof. We only prove (1) in the case $f\colon X = \mathbb{P}^n_{\mathbb{C}} \to \mathrm{Spec}(\mathbb{C})$ following Serre. The general case is reduced to this (using Chow's lemma, etc.). Now, we will use two "analytic" ingredients, which are show, e.g., using techniques from Hodge theory.

(1) If Y is a compact complex manifold and $\mathcal{F} \in \mathrm{Coh}_Y$, then $H^*(Y,\mathcal{F})$ is a finite dimensional $\mathbb{C}$-vector space.

(2) $R\Gamma(\mathbb{C}P^n, \mathcal{O}) \cong \mathbb{C}[0]$.

Now we do an induction on n. The short exact sequences $0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_{\mathbb{P}^{n-1}_{\mathbb{C}}} \to 0$ and $0 \to \mathcal{O}_{X^{\mathrm{an}}}(-1) \to \mathcal{O}_{X^{\mathrm{an}}} \to \mathcal{O}_{\mathbb{C}PP^{n-1}} \to 0$ show that the statement follows for the twist bundles $\mathcal{O}_X(k), k \in \mathbb{Z}$, if we know the statement for $\mathcal{O}_X$. But this case follows from the second analytic ingredient above. To pass to all coherent sheaves one uses a downward induction on i because both sides vanish for $i \gg 0$ (the RHS, e.g., by Theorem 4.35 or using Čech cohomology and Cartan's Theorem A). If $\mathcal{M} \in \mathrm{Coh}_X$, then $\mathcal{M}(k) := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(k)$ is globally generated for $k \gg 0$, i.e., there exists a surjection $\mathcal{O}_X(-k)^m \to \mathcal{M}$. Using the case of $\mathcal{O}_X(-k)$, the long exact sequence in cohomology and the induction on i settles (1).

Now, (1) implies fully faithfulness in (2) because

$$\mathrm{Hom}_X(\mathcal{E}, \mathcal{F}) = H^0(X, \mathcal{H}om_X(\mathcal{E}, \mathcal{F}))$$

for $\mathcal{E}, \mathcal{F} \in \mathrm{Coh}_X$ and similarly on X^{an}. To prove essential surjectivity one again reduces to $X = \mathbb{P}^n_{\mathbb{C}}$. Now, we'd like to see that each $\mathcal{M} \in \mathrm{Coh}_{X^{\mathrm{an}}}$ is globally generated after sufficiently many twisting by $\mathcal{O}_{X^{\mathrm{an}}}(k)$. Granting this, the claim follows by exactness of $(-)^{\mathrm{an}}$ as one can find a resolution $\mathcal{E}^{\mathrm{an}} \xrightarrow{\alpha} \mathcal{F}^{\mathrm{an}} \to \mathcal{M} \to 0$ with $\mathcal{E}, \mathcal{F}$ direct sums of $\mathcal{O}_X(k)'s$, and $\alpha = (\beta)^{\mathrm{an}}$ for some $\beta\colon \mathcal{E} \to \mathcal{F}$ by the proven fully faithfulness. Let $x \in X$ and let $E \cong \mathbb{P}^{n-1}_{\mathbb{C}}$ be a hyperplane through x. Then one for any $k \in \mathbb{Z}$ short exact sequences

$$0 \to \mathcal{G}(k-1) \to \mathcal{M}(k-1) \to \mathcal{H}(k) \to 0$$
$$0 \to \mathcal{H}(k) \to \mathcal{M}(k) \to \mathcal{M}(k) \otimes_{O_{X^{\mathrm{an}}}} \mathcal{O}_{E^{\mathrm{an}}} \to 0$$

for some $\mathcal{G}, \mathcal{H} \in \mathrm{Coh}_{X^{\mathrm{an}}}$. Namely, we can set $\mathcal{H} := \ker(\mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathcal{O}_E)$ and $\mathcal{G} := \ker(\mathcal{M} \to \mathcal{M}(1))$. The induction on n implies that for $k \gg 0$ the higher cohomology on X^{an} of $\mathcal{G}(k)$ and $\mathcal{M}(k) \otimes_{O_{X^{\mathrm{an}}}} \mathcal{O}_{E^{\mathrm{an}}}$ vanishes. Indeed, both sheaves have support on E^{an} and admit filtrations with gradeds given by analytic coherent sheaves of $\mathcal{O}_{E^{\mathrm{an}}}$-modules. By induction, these graded are algebraic and hence their higher cohomology vanishes for sufficiently high twists. We can conclude that for all $k \gg 0$ the map

$$H^i(X^{\mathrm{an}}, \mathcal{M}(k-1)) \to H^i(X^{\mathrm{an}}, \mathcal{H}(k))$$

is an isomorphism for any $i > 0$, and the map

$$H^1(X^{\mathrm{an}}, \mathcal{H}(k)) \to H^1(X^{\mathrm{an}}, \mathcal{M}(k))$$

is surjective. In particular,

$$\dim_{\mathbb{C}} H^1(X^{\mathrm{an}}, \mathcal{M}(k-1)) \geq \dim_{\mathbb{C}} H^1(X^{\mathrm{an}}, \mathcal{M}(k))$$

for $k \gg 0$. By finiteness of the dimensions $H^1(X^{\mathrm{an}}, \mathcal{M}(k))$, we can assume these dimensions are constant for $k \gg 0$. But then

$$H^1(X^{\mathrm{an}}, \mathcal{H}(k)) \to H^1(X^{\mathrm{an}}, \mathcal{M}(k))$$

is an isomorphism, and hence $H^0(X^{\mathrm{an}}, \mathcal{M}(k)) \to H^0(X^{\mathrm{an}}, \mathcal{M} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathcal{O}_{E^{\mathrm{an}}})$ is surjective. This implies (by induction on n and Nakayama) that $\mathcal{M}(k)$ is generated at x by global sections. By coherence, this will hold in a neighborhood of x and by compactness of $\mathbb{C}P^n$, we can find some $k \gg 0$ that works for all $x \in X^{\mathrm{an}}$. This finishes the proof. □

Exercise 3.52. Show that both assertions in Theorem 3.51 fail without the assumption of properness. *Hint: Consider a non-zero holomorphic function $g\colon \mathbb{C} \to \mathbb{C}$ with infinitely many zeros.*

Remark 3.53. Theorem 3.51 has several nice consequences. Assume that $X \to \mathrm{Spec}(\mathbb{C})$ is a proper morphism of schemes.

(1) $H^0(X, \mathcal{O}_X) \cong H^0(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$. In particular, X is connected if and only if X^{an} is connected. If X is projective, then this settles the remaining statement for Proposition 3.39.
(2) The functor $(-)^{\mathrm{an}}\colon \mathrm{Coh}_X \to \mathrm{Coh}_{X^{\mathrm{an}}}$ induces a bijection between coherent ideal sheaves $\mathcal{O}_X$ and $\mathcal{O}_{X^{\mathrm{an}}}$. In particular, each closed analytic subspace of X^{an} is algebraic. This has the following concrete consequence (Chow's theorem): If $Y \subseteq \mathbb{C}P^n$ is a closed analytic subset, then Y is the vanishing locus of finitely many homogeneous polynomials.
(3) The functor $X \mapsto X^{\mathrm{an}}$ from *proper* schemes over $\mathbb{C}$ to complex analytic spaces is fully faithful. Indeed, note that for X, Y proper of $\mathbb{C}$ the set $\mathrm{Hom}_{\mathbb{C}}(Y, X)$ identifies via associating to a morphism its graph with the set of closed subschemes $\Gamma \subseteq Y \times_{\mathrm{Spec}(\mathbb{C})} X$ such that the projection $\Gamma \to Y$ is an isomorphism. By the previous point and Proposition 3.41 this set is in bijection with its analog for $X^{\mathrm{an}}, Y^{\mathrm{an}}$, hence with $\mathrm{Hom}_{\mathbb{C}}(Y^{\mathrm{an}}, X^{\mathrm{an}})$.
(4) If $X \to \mathrm{Spec}(\mathbb{C})$ is proper, then $H^1(X^{\mathrm{an}}, \mathcal{O}^*_{X^{\mathrm{an}}}) \cong H^1(X, \mathcal{O}^*_X)$ because both identify with isomorphim classes of invertible coherent sheaves.

Exercise 3.54. Use GAGA and Lemma 3.49 to prove the following: If X, Y are abelian varieties over $\mathrm{Spec}(\mathbb{C})$, then the $\mathbb{Z}$-module of morphisms $Y \to X$ of group schemes, is finite free. *Remark: Using different arguments, the same statement can be proven for $\mathbb{C}$ replaced by any field.*

We stop here with our discussion of analytification and come back to the developement of cohomology theories for schemes.

4. Sheaf cohomology for topological spaces

4.1. **Betti cohomology and Serre's objection.** Using analytification we can define our first cohomology theory for schemes over $\mathrm{Spec}(\mathbb{C})$. Fix an abelian group G. Then we can define the contravariant functor

$$H^*_{\mathrm{Betti}}(-,G)\colon (\mathrm{Sch}^{\text{loc. of finite type}}/\mathbb{C}) \to (\mathrm{Ab}),\ X \mapsto H^*(X^{\mathrm{an}}, \underline{G}),$$

which is a perfectly well-behaved cohomology theory (called Betti cohomology) for schemes which are locally of finite type over $\mathbb{C}$. If k is any algebraically closed field and $G = \mathbb{Z}/n$ with n invertible in k, then étale cohomology will provide a functor

$$H^*_{\text{ét}}(-,\mathbb{Z}/n)\colon (\mathrm{Sch}^{\text{loc. of finite type}}/k) \to (\mathrm{Ab}),\ X \mapsto H^*_{\text{ét}}(X, \mathbb{Z}/n),$$

which will have properties very similar to $H^*_{\mathrm{Betti}}(-,\mathbb{Z}/n)$, for example $H^i_{\text{ét}}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}/n) \cong \mathbb{Z}/n$ for $i = 0, 2, \ldots, 2n$ and zero otherwise. However, étale cohomology will not yield a general theory with $\mathbb{Z}$ or $\mathbb{Q}$-coefficients, by a famous observation of Serre.

Lemma 4.2 (Serre's objection). *Let k be an algebraically closed field of characteristic $p > 0$. Then there cannot exists a cohomology theory*

$$H^*(-)\colon \{\textit{projective, smooth schemes over } k\} \to \{\textit{graded vector spaces over } \mathbb{R}\}$$

satisfying the Künneth formula such that $H^1(-)$ sends each elliptic curve $E \to \mathrm{Spec}(k)$ to an $\mathbb{R}$-vector space of dimension 2.

If $E \to \mathrm{Spec}(\mathbb{C})$ is an elliptic curve, then $E^{\mathrm{an}} \cong \mathbb{C}/\Lambda$ for some lattice $\Lambda \subseteq \mathbb{C}$, cf. Lemma 3.49. Hence, $H^1_{\mathrm{Betti}}(E, G) \cong G^2$ for any abelian group G, and thus we'd like this to happen also for our cohomology theory for schemes over k. Moreover, the Künneth formula

$$H^*(T \times S, \mathbb{R}) \cong H^*(T, \mathbb{R}) \otimes_{\mathbb{R}} H^*(S, \mathbb{R})$$

holds for topological spaces T, S, and thus we'd like it to hold for a cohomology theory over k as well.

Proof. If $E \to \mathrm{Spec}(k)$ is a supersingular elliptic curve, i.e., $E[p](k) = \{0\}$, then it is classical that $D := \mathrm{End}_k(E)$ is a subring of a quaterion algebra over $\mathbb{Q}$, such that $D \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{H}$ is isomorphic to Hamiltons quaternion division algebra. As $H^1(-)$ is a functor, the $\mathbb{R}$-vector space $H^1(E)$ will be equipped with an action of the multiplicative monoid D. As H^* is assumed to satisfy the Künneth formula, this action makes $H^1(E)$ into a *module* for D. As $H^1(E)$ is an $\mathbb{R}$-vector space, we can extend scalars and get a $\mathbb{R} \otimes_{\mathbb{Z}} D \cong \mathbb{H}$-module structure on $H^1(E)$. But $\mathbb{H}$ is a division algebra of dimension 4 over $\mathbb{R}$ and hence cannot act on a 2-dimensional $\mathbb{R}$-vector space! □

Lemma 4.2 motivates the assumption that n is invertible in k. Namely, $\mathbb{Z}/n \otimes_{\mathbb{Z}} D$ won't be a division algebra anymore and hence can act on a finite free $\mathbb{Z}/n$ of rank 2.

Before constructing $H^*_{\text{ét}}(-, \mathbb{Z}/n)$ we will develope sheaf cohomology on topological spaces, and thus $H^*_{\mathrm{Betti}}(-, G)$, in more detail. In particular, we want to show that sheaf cohomology satisfies as good properties as singular cohomology, maybe even better.

4.3. **De Rham cohomology for real manifolds.** Let T be a (smooth) real manifold. Then the sheaf cohomology $H^*(X, \underline{\mathbb{R}})$ can conveniently be represented by smooth differential forms. For $k \geq 0$ let

$$\mathcal{A}^k_{\mathbb{R},T}$$

be the sheaf of (smooth $\mathbb{R}$-valued) differential k-forms on T, e.g, $\mathcal{A}^0_{\mathbb{R},T} = \mathcal{C}^\infty_{\mathbb{R},T}$ is the sheaf of $\mathcal{C}^\infty$-functions on T. Let

$$d\colon \mathcal{A}^k_{\mathbb{R},T} \to \mathcal{A}^{k+1}_{\mathbb{R},T}$$

be the exterior differential, i.e., the unique map of sheaves, which sends a differential k-form $\omega = f(x_1, \ldots, n)dx_{j_1} \wedge \ldots \wedge dx_{j_k}$ in coordinates to

$$d\omega := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_k}.$$

Lemma 4.4 (Poincaré lemma, cf. [18, Lemma 7.11]). *Let T be a real manifold. Then the de Rham complex*

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}^0_{\mathbb{R},T} \xrightarrow{d} \mathcal{A}^1_{\mathbb{R},T} \xrightarrow{d} \ldots$$

is exact.

Proof. It suffices to check exactness for the point 0 in the open unit disc $T \subseteq \mathbb{R}^n$. In this case the de Rham complex $\mathcal{A}^\bullet_{\mathbb{R},T}(T)$ is actually homotopic to 0. Namely, define the map

$$h^k \colon \mathcal{A}^k_{\mathbb{R},T} \to \mathcal{A}^{k-1}_{\mathbb{R},T}$$

sending $\omega = f(x_1, \ldots, x_n) dx_{j_1} \wedge \ldots \wedge dx_{j_k}, j_1 < \ldots < j_j$ to the differential

$$x_{j_1}(\int\limits_0^1 f(0, \ldots, 0, tx_{j_1}, x_{j_1+1}, \ldots, x_n)dt) dx_{j_2} \wedge \ldots \wedge dx_{j_k}.$$

Then a calculation shows that the collection $h^k, k \geq 0$ defines a homotopy between the identity and zero on $\mathcal{A}^\bullet_{\mathbb{R},T}(T)$. □

Now the sheaves $\mathcal{A}^k_{\mathbb{R},T}$ flasque for a very simple reason as shown by the next lemma.

Lemma 4.5. *Assume that T is a paracompact, Hausdorff real manifold.*[20] *Let $\mathcal{F}$ be a sheaf of $\mathcal{R} := \mathcal{C}^\infty_{\mathbb{R},T}$-modules. Then $\mathcal{F}$ is acylic. In particular, $R\Gamma(T, \underline{\mathbb{R}}) \cong \Gamma(T, \mathcal{A}^\bullet_{\mathbb{R},T})$ by Lemma 4.4 as each term in $\mathcal{A}^\bullet_{\mathbb{R},T}$ admits a module structure under $\mathcal{C}^\infty_{\mathbb{R},T}$.*

The proof is taken from [29, Proposition 4.36].

Proof. The sheaf $\mathcal{R}$ of rings on the paracompact real manifold T has the following critical property[21]: If $T = \bigcup\limits_{i \in I} U_i$ is a open cover, then there exists $f_i \in \mathcal{R}(T)$ such that $\mathrm{supp}(f_i) \subseteq U_i$ and $\sum\limits_{i \in I} f_i = 1$, where the sum is assumed to be locally finite. This property implies that $H^k(U, \mathcal{F}) = 0$ for $k > 0$ and any sheaf $\mathcal{F}$ of $\mathcal{R}$-modules on T. Indeed, let $\mathcal{F} \to \mathcal{I}^\bullet$ be a resolution of $\mathcal{F}$ by injective $\mathcal{R}$-modules. Assume that $\alpha \in \mathcal{I}^k(T)$ is a cocycle. If $k > 0$, then there exists an open cover $T = \bigcup\limits_{i \in I} U_i$ of T and sections $\beta_i \in \mathcal{I}^{k-1}(U_i)$ such that $d\beta_i = \alpha_{|U_i}$. Let $f_i \in \mathcal{R}(T)$ as above. Then $f_i\beta_i \in \mathcal{I}^{k-1}(U_i)$ has support in a closed subset of U_i and can therefore be extended by 0 to a section $\gamma_i \in \mathcal{I}^{k-1}(T)$. Set $\gamma := \sum\limits_{i \in I} \gamma_i \in \mathcal{I}^{k-1}(T)$, where the sum is locally finite by assumption. Now, $d\gamma = \alpha$ because this can be checked locally and $\sum\limits_{i \in I} f_i = 1$. □

Remark 4.6. The de Rham complex $\mathcal{A}^\bullet_{\mathbb{R},T}$ is the prototypical example of a sheaf of differential graded algebras, cf. [Stacks, Tag 061V]. Namely, the $\wedge$-product endows $\bigoplus\limits_{k \geq 0} \mathcal{A}^k_{\mathbb{R},T}$ with the structure of an $\mathbb{R}$-algebra and the differential d relates to $\wedge$ via the equation

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^i \omega \wedge d(\eta)$$

for $\omega \in \mathcal{A}^i_{\mathbb{R},T}, \eta \in \mathcal{A}^j_{\mathbb{R},T}$. Phrased differently, the $\wedge$-product yields a morphism $\mathcal{A}^\bullet_{\mathbb{R},T} \otimes_{\mathbb{R}} \mathcal{A}^\bullet_{\mathbb{R},T} \to \mathcal{A}^\bullet_{\mathbb{R},T}$ of complexes satisfying associativity. Clearly, the resolution $\mathbb{R} \to \mathcal{A}^\bullet_{\mathbb{R},T}$ is a morphism of sheaves of differential algebras. We can conclude that the $\wedge$-product computes the $\cup$-product in cohomology, i.e., the diagram

$$\begin{array}{ccc} \Gamma(T, \mathcal{A}^\bullet_{\mathbb{R},T}) \otimes_{\mathbb{R}} \Gamma(T, \mathcal{A}^\bullet_{\mathbb{R},T}) & \longrightarrow & \Gamma(T, \mathcal{A}^\bullet_{\mathbb{R},T}) \\ \downarrow & & \downarrow \\ R\Gamma(T, \underline{\mathbb{R}}) \otimes^L_{\mathbb{R}} R\Gamma(T, \underline{\mathbb{R}}) & \xrightarrow{\cup} & R\Gamma(T, \underline{\mathbb{R}}) \end{array}$$

commutes, cf. [Stacks, Tag 0FP3]. Similarly, we can see that the cup-product for the singular cochain complex makes the resolution $\underline{\mathbb{Z}} \to \mathcal{C}^\bullet_{\mathrm{sing},T}$ from Theorem 2.16 a quasi-isomorphism of sheaves of differential graded algebras. Hence, the cup-product on $R\Gamma(T, \underline{\mathbb{Z}})$ identifies with the product in singular cohomology.

[20]Often this is part of the definition. We will assume this often without mentioning.

[21]This property is also called existence of partitions of unity.

4.7. **Excision for singular cohomology.** In order to speak about excision for sheaf cohomology, we first have to find relative sheaf cohomology groups $H^*(T, A; \mathcal{F})$ for a topological space T, a subspace $A \subseteq T$ and $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T)$, fitting into a long exact sequence

$$\ldots \to H^n(T, A; \mathcal{F}) \to H^n(T, \mathcal{F}) \to H^n(A, \mathcal{F}) \to \ldots$$

Usually the most interesting case is that for $A \subseteq T$ open or closed. In this section, we focus on the case that $i\colon A \to T$ is a *closed* immersion, and thus its complement $j\colon W := T \setminus A \to T$ is an open immersion.

Definition 4.8. The functor

$$j_!\colon \mathrm{Sh}_{\mathrm{Ab}}(W) \to \mathrm{Sh}_{\mathrm{Ab}}(T)$$

is defined by sending $\mathcal{F}$ to the sheaf

$$U \subseteq T \text{ open } \mapsto j_!\mathcal{F}(U) := \{s \in \mathcal{F}(W \cap U) \mid \mathrm{supp}(s) \text{ is closed in } U\}.$$

Here, the support of s is the closed set of points $t \in W$, where the germ $s_t \in \mathcal{F}_t$ of s vanishes. The support $\mathrm{supp}(s)$ is always closed in W.

Remark 4.9. (1) Clearly, there exists a natural transformation $j_! \to j_*$ of functors.
(2) If $j'\colon W' \to W$ is another open immersion, then $j_! \circ j'_! \cong (j \circ j')_!$ as follows directly from the definition.
(3) The natural map $j^* j_! \mathcal{F} \to j^* j_*(\mathcal{F}) \cong \mathcal{F}$ is an isomorphism because for $U \subseteq W$ open the support of $s \in \mathcal{F}(U)$ is closed in U.
(4) If $t \in T$ is a point, then the stalk $(j_!\mathcal{F})_t$ is $\mathcal{F}_t$ if $t \in W$ and 0 if $t \in T \setminus W$. Indeed, the first case follows from $j^* j_! \mathcal{F} \cong \mathcal{F}$ and the second from the fact that if $s \in j_!(\mathcal{F})(U)$ is a section, then $s_{|U \setminus \mathrm{supp}(s)} = 0$.

Example 4.10. Let T be a real manifold and $j\colon U \to T$ the inclusion of an open subset with complement Z. Then $j_!(\underline{\mathbb{R}}) \cong j_!(\mathcal{A}^\bullet_{\mathbb{R},U})$. Now, $j_!(\mathcal{A}^k_{\mathbb{R},U})$ identifies with differential k-forms ω on U, which have "compact supports towards $T \setminus Z$", e.g., if T is compact, then ω simply has compact support. By Lemma 4.5 we can conclude that $j_!(\mathcal{A}^\bullet_{\mathbb{R},U})$ is a resolution o $j_!(\mathbb{R})$ by acyclic sheaves as it is a complex with terms given by modules under $\mathcal{C}^\infty_{\mathbb{R},T}$. Hence, the relative cohomology $H^*(T, Z; \underline{\mathbb{R}})$, which is the cohomology of

$$R\Gamma(T, j_!(\mathbb{R})) \cong \Gamma(T, j_!(\mathcal{A}^\bullet_{\mathbb{R},U})),$$

can be handled rather explicitly. Particularly, if T is compact, then $\Gamma(T, j_!(\mathcal{A}^\bullet_{\mathbb{R},U})) = \mathcal{A}^\bullet_{\mathbb{R},c}(U)$ is calculated via the complex of differential forms on U with compact support.

The functor $j_!$ is left adjoint to the functor j^* as we now prove.

Lemma 4.11. (1) *The functor $j_!\colon \mathrm{Sh}_{\mathrm{Ab}}(W) \to \mathrm{Sh}_{\mathrm{Ab}}(T)$ is left adjoint to the functor $j^*\colon \mathrm{Sh}_{\mathrm{Ab}}(T) \to \mathrm{Sh}_{\mathrm{Ab}}(W)$.*
(2) *The unit $\mathcal{G} \to j^* j_!(\mathcal{G})$ is an isomorphism for any $\mathcal{G} \in \mathrm{Sh}_{\mathrm{Ab}}(W)$.*
(3) *For any $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T)$ the "excision sequence"*

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

is exact. In particular, for any $K \in D(T, \underline{\mathbb{Z}})$ there exists the natural "excision triangle"

$$j_! j^* K \to K \to i_* i^* K \to j_! j^* K[1].$$

Proof. We define the unit of the adjunction as the inverse of the natural isomorphism $j^* j_! \mathcal{G} \to \mathcal{G}$ constructed in Remark 4.9 for $\mathcal{G} \in \mathrm{Sh}_{\mathrm{Ab}}(W)$. Given $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T)$ we define the counit $j_! j^* \mathcal{F} \to \mathcal{F}$ via the map

$$j_! j^* \mathcal{F}(U) = \{s \in \mathcal{F}(U \subseteq W) \mid \mathrm{supp}(s) \text{ is closed in } U\} \to \mathcal{F}(U)$$

for $U \subseteq T$ open, by extending s by 0 to a section in $\mathcal{F}(U)$ (which is possible as one can glue using the open cover $U \cap W$, $U \setminus \mathrm{supp}(s)$ of U). The triangle identities for the adjunction can be checked on stalks, where they are easy. Also, the exactness of the excision sequence can be checked on stalks. □

Remark 4.12. In the lecture a functor $j_!$ (with the same definition) was introduced for $j\colon W \to T$ only assumed to be locally closed. Then $j_!$ admits an adjoint $j^!$ given by sections with support in W. If $i\colon A = T \setminus W \subseteq T$ is additionally assumed to be locally closed, we get functors $i_!, i^!$ and the excision triangle can be generalized to a triangle

$$i_! R i^! K \to K \to R j_* j^* K \to i_! R i^! K[1]$$

for any $K \in \mathcal{D}(T, \underline{\mathbb{Z}})$. In fact, the morphisms $i_! R i^! K \to K, K \to R j_* j^* K$ are given by the counit/unit respectively. Checking that they form a distinguished triangle can be done on stalks.

If $t \in T \setminus \overline{W}$, then $(Rj_*j^*K)_t = 0$, and $(i_!Ri^!K)_t \cong K$. From here, one reduces to the case that $T = \overline{W}$, in which case j is an open immersion. Then one argues via deriving the exact sequence

$$0 \to i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F},$$

in which the last morphisms is surjective if $\mathcal{F}$ is injective.

Now we can define the relative cohomology groups.

Definition 4.13. For $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T)$ and $i\colon A \to T$ a closed immersion, we define the relative sheaf cohomology as

$$H^*(T, A; \mathcal{F}) := \mathcal{H}^* R\Gamma(T, j_!Rj^!\mathcal{F}).$$

From Lemma 4.11 we can conclude the existence of a distinguished triangle

$$R\Gamma(T, j_!j^*\mathcal{F}) \to R\Gamma(T, \mathcal{F}) \to R\Gamma(T, i_*i^*\mathcal{F}) \cong R\Gamma(A, i^*\mathcal{F}) \to R\Gamma(T, j_!j^*\mathcal{F})[1]$$

and a long exact sequence

$$\ldots \to H^n(T, A; \mathcal{F}) \to H^n(T, \mathcal{F}) \to H^n(A, i^*\mathcal{F}) \to \ldots$$

Our next aim is to discuss some concrete examples for relative sheaf cohomology groups. We will see that this naturally leads to the proper base change theorem.

4.14. **The proper base change theorem in topology.** Assume that T is a topological space and $i\colon A \subseteq T$ a *closed* subspace. Let $j\colon W \to T$ be its open complement. Moreover, assume that T is compact. Motivated by Example 4.10 we define the "compactly supported cohomology"

$$H^*_c(W, \mathcal{G}) := H^*(T, j_!\mathcal{G})$$

for $\mathcal{G} \in \mathrm{Sh}_{\mathrm{Ab}}(W)$ and conclude the existence of a long exact sequence

$$\ldots \to H^i_c(W, j^*\mathcal{F}) \to H^i(T, \mathcal{F}) \to H^i(A, \mathcal{F}) \to \ldots$$

for any $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T)$ by Definition 4.13.

Example 4.15. As a concrete example take $T = \mathbb{C}P^n$ and $i\colon A := \mathbb{C}P^{n-1} \to \mathbb{C}P^n$, $(x_0 : \ldots : x_{n-1}) \mapsto (x_0 : \ldots : x_{n-1} : 0)$. Then $j\colon W \cong \mathbb{C}^n \to \mathbb{C}P^n$ is a standard open set. We can conclude that for any abelian group G we have $H^i_c(\mathbb{C}^n, \underline{G}) = 0$ if $i \neq 2n$ and G if $i = 2n$.

Note that the notation $H^*_c(W, \mathcal{G})$ is not justified at the moment - its definition depends (a priori) on the compactification T of W! If (T, A) is a good pair, then excision in singular cohomology implies by Proposition 2.6 that

$$H^*_{\mathrm{sing}}(T, A; G) \cong H^*_{\mathrm{sing}}(T/A, *; G).$$

As we assumed T is compact and A closed, the map

$$f\colon T \to T/A$$

is proper. Moreover, f induces an isomorphism $W \to (T/A) \setminus A/A$. Now, T/A identifies with the one-point compactification of W and hence the compactly supported cohomology (with coefficients in $\underline{G}$) is indeed independent of the compactification T of W.

Let us try to develop a sheaf theoretic proof of this independence.

Let $j'\colon W \to T'$ be another open immersion with T' compact, and assume that there exist a map $f\colon T' \to T$, which restricts to an isomorphism $j'(W) \to j(W)$. Moreover, assume that $f^{-1}(j(W)) = j'(W)$. As T', T are assumed to be compact the map f is proper. Let $\mathcal{G} \in \mathrm{Sh}_{\mathrm{Ab}}(W)$. Let us apply the excision triangle for j to $Rf_*(j'_!(\mathcal{G}))$. This yields the distinguished triangle

$$j_!\mathcal{G} \cong j_!j^*Rf_*(j'_!(\mathcal{G})) \to Rf_*(j'_!(\mathcal{G})) \to i_*i^*Rf_*(j'_!(\mathcal{G})).$$

The first isomorphism comes from the isomorphisms

$$j^*(Rf_*(j'_!(\mathcal{G}))) \cong (j')^*j'_!(\mathcal{G}) \cong \mathcal{G}$$

by our assumptions on the map f and the fact that Rf_* commutes with restriction. The desired independence of the compactly supported cohomology on T means that we'd like the map

$$R\Gamma(T, j_!\mathcal{G}) \to R\Gamma(T', j'_!\mathcal{G}) \cong R\Gamma(T, Rf_*(j'_!(\mathcal{G})))$$

is an isomorphism in $\mathcal{D}(\mathbb{Z})$, or equivalently by the distinguished triangle above, that

$$0 = R\Gamma(T, i_*(i^*Rf_*(j'_!(\mathcal{G})))) \cong R\Gamma(A, i^*Rf_*(j'_!\mathcal{G})).$$

If T is the one-point compactification of W, then $A = \{t\}$ is one point, and this vanishing is equivalent to the vanishing of the stalk

$$(Rf_*(j'_!(\mathcal{G})))_t.$$

Under the assumption $A = \{t\}$, this last vanishing is very plausible as the sheaf $j'_!(\mathcal{G}))_{f^{-1}(t)}$ vanishes on $f^{-1}(t) = T' \setminus j'(W)$.

The proper base change theorem now implies that our wish is indeed true.

Theorem 4.16 (Proper base change). *Let $f\colon T' \to T$ be any proper map of topological spaces, cf. Lemma 3.17. Let $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T')$. Then for any $t \in T$ the natural map*

$$(Rf_*(\mathcal{F}))_t \to R\Gamma(f^{-1}(t), \mathcal{F}_{|f^{-1}(t)})$$

is an isomorphism in $\mathcal{D}(\mathbb{Z})$. (Here, $\mathcal{F}_{|f^{-1}(t)} = i_t^*\mathcal{F}$ for the inclusion $i_t\colon f^{-1}(t) \to T'$.)

Before proving this theorem, let us make some remarks.

Remark 4.17. (1) The properness of f is essential. For example, if $f\colon T' = \mathbb{C}^* \to T = \mathbb{C}$ is the open inclusion, and $\mathcal{F} = \underline{\mathbb{Z}}$, then for $t = 0 \in T$ we get

$$f_*(\underline{\mathbb{Z}}_{T'}) \cong \mathbb{Z}_T,$$

because for each open ball D around 0, the space $D \setminus \{0\}$ is connected. But $f^{-1}(t)$ is empty, hence $R\Gamma(f^{-1}(t), \mathcal{F}_{|f^{-1}(t)}) = 0$. Note that in this case the fibers of f are compact.

(2) The proper base change theorem extends (rather formally) to any $K \in D^+(T', \underline{\mathbb{Z}})$ instead of $\mathcal{F}$. Indeed, for a fixed $i \in \mathbb{Z}$, the map

$$\mathcal{H}^i((Rf_*(K))_t) \to H^i(f^{-1}(t), K_{|f^{-1}(t)})$$

only depends on the truncation $\tau_{\leq i}K$ and for a bounded complex one checks that it is an isomorphism by reducing to Theorem 4.16 using the exactness of both functors $\mathcal{D}(T', \underline{\mathbb{Z}}) \to \mathcal{D}(\mathbb{Z})$.

(3) Consider a cartesian square

$$\begin{array}{ccc} S' & \xrightarrow{g'} & T' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{g} & T \end{array}$$

of topological spaces and assume that f is proper. From Theorem 4.16 one can deduce that for any $K \in D^+(T', \underline{\mathbb{Z}})$ the natural map[22]

$$g^*Rf_*(K) \to Rf'_*(g'^{,*}K)$$

is an isomorphism. Indeed, this can be checked at the stalks at points $s \in S$, and then one applies Theorem 4.16 to f and its base change f' (which is again proper by Lemma 3.17), cf. [Stacks, Tag 09V6].

We now turn to the proof of Theorem 4.16. For simplicity, we will only present the proof under the (usally harmless) assumption that T', T are locally compact. The general case can be found in [Stacks, Tag 09V4].

Our proof will rest on three lemmata. The first is this one.

Lemma 4.18. *Let $f\colon T' \to T$ be a closed map, e.g., f proper, and let $t \in T$. If $U \subseteq T'$ is an open neighborhood of $f^{-1}(t)$, then there exists an open neighborhood V of t, such that $f^{-1}(V) \subseteq U$.*

Proof. By assumption on f the set $A := f(T' \setminus U) \subseteq T$ is closed. As $f^{-1}(t) \subseteq U$, we get that $t \notin A$. Now we can set $V := T \setminus A$. □

The second lemma is this.

Lemma 4.19. *Let T be a locally compact Hausdorff space and let $i\colon Z \to T$ be a the inclusion of a compact subset. For any $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T)$ the natural maps*

$$\varinjlim_{Z \subseteq U \subseteq T \text{ open}} Rj_{U,*}j_U^*\mathcal{F} \to i_*i^*\mathcal{F}, \qquad \varinjlim_{Z \subseteq U \subseteq T \text{ open}} j_{U,*}j_U^*\mathcal{F} \to i_*i^*\mathcal{F}$$

are isomorphisms. (Here, $j_U\colon U \to T$ denotes the associated open immersion.)

Proof. If $t \in T$ is a point in Z and $Z \subseteq U$ an open neighborhood, then

$$(Rj_{U,*}j_U^*\mathcal{F})_t \cong (j_{U,*}j_U^*\mathcal{F})_t \cong \mathcal{F}_t \cong i_*i^*\mathcal{F}$$

as $t \in U$. If $t \in T \setminus Z$, then by compactness of Z there exists an open neighborhood $Z \subseteq U$ of Z such that $t \notin \overline{U}$. Then

$$(Rj_{U,*}j_U^*\mathcal{F})_t \cong (j_{U,*}j_U^*\mathcal{F})_t \cong 0 \cong (i_*i^*\mathcal{F})_t.$$

This finishes the proof. □

[22] adjoint to the map $Rf_*(K) \to Rg_*Rf_*(g'^{,*}K) \cong Rf_*(Rg'_*g'^{,*}K)$ induced by the unit $K \to Rg'_*g'^{,*}K$

The third one is more difficult.

Lemma 4.20. *Let T be a compact topological space*[23] *and let $\mathcal{F}_i \in \mathrm{Sh}_{\mathrm{Ab}}(T), i \in I$, be a filtered system of abelian sheaves with colimit $\mathcal{F}$. Then for any $n \geq 0$ the natural map*

$$\varinjlim_{i\in I} H^n(T,\mathcal{F}_i) \to H^n(T,\mathcal{F})$$

is an isomorphism.

Proof. We will prove the statement by induction on n. Assume $n = 0$. Let $s \in \Gamma(T,\mathcal{F}_i)$ be a section vanishing in $\Gamma(T,\mathcal{F})$. For each point $t \in T$ there exists then some $i \leq i'_t$ and an open subset $U_t \subseteq T$ such that $s_{|U_t}$ vanishes in $\mathcal{F}_{i'_t}(U_t)$. As T is covered by finitely many of these U_t, we can conclude that s restricts to 0 in $\Gamma(T,\mathcal{F}_{i'})$ for i' sufficiently large. Now let $s \in \Gamma(T,\mathcal{F})$ be a section. For any point $t \in T$ there exists then some i and some open neighborhood U_t of t such that s is the image of some $s_t \in \mathcal{F}_i(U_t)$. Let $Z_t \subseteq U_t$ be some compact neighborhood of t. There exists finitely many $t_1,\ldots,t_r \in T$ such that the interiors of Z_t cover T. By the proven injectivity and the compactness of $Z_{t_l} \cap Z_{t_k}$ we may now find some j large enough such that $s_{t_l} = s_{t_k}$ in $\Gamma(Z_{t_l} \cap Z_{t_k}, \mathcal{F}_{j,|Z_{t_l}\cap Z_{t_k}})$. But then we can glue the section $s_{t_l,|\mathring{Z}_{t_l}}$ to a section in $\Gamma(T,\mathcal{F}_j)$, which maps to s.

Now let's assume that the statement is true for any $m \leq n$. Using a functorial injective resolution and the case $n = 0$ the claim reduces to the case that each $\mathcal{F}_i$ is an injective abelian sheaf. In this case, we have to check that $\mathcal{F} = \varinjlim_{i\in I} \mathcal{F}_i$ is acyclic. By induction we know that $\mathcal{F}$ is n-acyclic, i.e., $H^m(T,\mathcal{F}) = 0$ for $0 < m < n$. Let $a \in H^{n+1}(T,\mathcal{F})$ be a class. For each point $t \in T$ there exists an open neighborhood $U_t \subseteq T$ such that $a_{|U_t} = 0$. As T is locally compact we can find some compact neighborhood $Z_t \subseteq U_t$ of t. As T is compact, there exists finitely many of such compact neighborhoods $Z_1,\ldots,Z_r$ such that $T = \bigcup_{j=1}^{r} Z_j$. We may also assume that already the interiors of the Z_j cover T. Let $i_j \colon Z_j \to T$ be the closed immersion. Consider the short exact sequence

$$0 \to \mathcal{F} \to \bigoplus_{j=1}^{r} i_{j,*}i_j^*\mathcal{F} \to Q \to 0$$

of sheaves on T (the exactness on the left follows from the sheaf axiom as the interiors of the Z_j cover T). By the long exact sequence

$$\ldots \to \bigoplus_{i=1}^{r} H^n(T, i_{j,*}i_j^*\mathcal{F}) \to H^n(T,\mathcal{Q}) \to H^{n+1}(T,\mathcal{F}) \to \bigoplus_{i=1}^{r} H^n(T, i_{j,*}i_j^*\mathcal{F}) \to \ldots$$

and the fact $a_{|Z_j} = 0$ (by construction of the Z_j), it suffices to see that the map $H^n(T,\mathcal{Q}) \to H^{n+1}(T,\mathcal{F})$ vanishes. Using Lemma 4.19 we can write $i_{j,*}i_j^*\mathcal{F} = \varinjlim_{i\in I} i_{j,*}i_j^*\mathcal{F}_i$ (looking at stalks it is clear that $i_{j,*}$ commutes with colimits) as a colimits of sheaves of the form $K_V := j_{V,*}j_V^*\mathcal{F}_i$ for $j_V \colon V \to T$ the inclusion of an open subset. Now, each K_V is an injective sheaf. Hence, we can conclude that $\bigoplus_{i=1}^{r} i_{j,*}i_j^*\mathcal{F}$ a colimit of injective sheaves. As cokernels of injective sheaves are again injective we also see that Q is a colimit of injective sheaves. If $n > 0$ we can imply the induction hypthesis and conclude the proof. If $n = 0$ then by the above discussion and the proven case $n = 0$ the map $\Gamma(T, \bigoplus_{j=1}^{r} i_{j,*}i_j^*\mathcal{F}) \to \Gamma(T,Q)$ is a filtered colimt of split surjections, and hence again surjective. [24] □

Now we can prove Theorem 4.16.

Proof of Theorem 4.16 in the case T',T locally compact. Note that the statement only depends on a neighborhood of $t \in T$. As we assume that T is locally compact we may therefore pass to a compact neighborhood and assume that T is compact. As f is proper, this implies that T' is compact, cf. Lemma 3.17. For an open neighborhood $U \subseteq T'$ of $Z := f^{-1}(t)$ let $j_U \colon U \to T'$ be the open inclusion and

$$K_U := Rj_{U,*}j_U^*\mathcal{F}.$$

If $U' \subseteq U$ is another open neighborhood of Z, then we get a natural morphism

$$K_U \to K_{U'}.$$

[23] This includes Hausdorff.

[24] We thank Jonas Walter for spotting some previous inaccuracies in the proof.

Using an injective resolution $\mathcal{I}^\bullet$ of $\mathcal{F}$ we see that $\{K_U\}_{Z\subseteq U} \cong \{j_{U,*}(\mathcal{I}^\bullet_{|U}\}$ is a filtered system of complexes of sheaves on T', which is uniformly bounded to the right of 0. Using dévissage in complexes we can deduce from Lemma 4.20 that the natural map

$$\varinjlim_{Z\subseteq U} H^*(T', K_U) \to H^*(T', \varinjlim_{Z\subseteq U} K_U)$$

is an isomorphism. Let $i\colon Z \to T'$ be the closed immersion. The natural restriction map $\varinjlim_{Z\subseteq U} K_U \cong i_*i^*\mathcal{F}$ is a quasi-isomorphism (or isomorphism in $\mathcal{D}(T', \underline{\mathbb{Z}})$) by Lemma 4.19. We can conclude that

$$H^*(T', \varinjlim_{Z\subseteq U} K_U) \cong H^*(T', i_*i^*\mathcal{F}) \cong H^*(Z, i^*).$$

By Lemma 4.18 the neighborhoods of Z given by $f^{-1}(V)$ with $V \subseteq T$ an open neighborhood of t are cofinal. Hence,

$$\varinjlim_{Z\subseteq U} H^*(T', K_U) \cong \varinjlim_{t\in V} H^*(T', K_{f^{-1}(V)}).$$

Now,

$$H^*(T', K_{f^{-1}(V)}) \cong H^*(f^{-1}(V), \mathcal{F}_{f^{-1}(V)})$$

and thus the colimit is, more or less by definition, equal to the stalk

$$(Rf_*(\mathcal{F}))_t.$$

This finishes the proof. □

4.21. **Consequences of the proper base change theorem.** Having the proper base change theorem at our disposal we now make the following definition.

Definition 4.22. Let T be a locally compact Hausdorff space and $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T)$.

(1) We define the compactly supported cohomology of $\mathcal{F}$ as

$$H^*_c(T, \mathcal{F}) := H^*(\overline{T}, j_!\mathcal{F}),$$

where $j\colon T \to \overline{T}$ is any open immersion with $\overline{T}$ a compact Hausdorff space.

(2) Similarly, we define

$$R\Gamma_c(T, \mathcal{F}) := R\Gamma(\overline{T}, j_!\mathcal{F}) \in \mathcal{D}(\mathbb{Z}).$$

(3) More generally, if $f\colon T \to S$ is a "compactifiable"[25] map of locally compact spaces, i.e, f can be factored into $T \xrightarrow{j} T' \xrightarrow{f'} S$ with j an open immersion and f' proper, then we define the "exceptional" pushforward

$$Rf_! := Rf'_* \circ j_!\colon \mathcal{D}(T, \underline{\mathbb{Z}}) \to \mathcal{D}(S, \underline{\mathbb{Z}}).$$

A priori, $Rf_!$ depends on the choice of "relative" compactification T'.

Exercise 4.23. Let $f\colon T \to S$ be a compactifiable map of locally compact spaces.

(1) Show that the category $\mathcal{C}$ of factorizations $\{T \xrightarrow{j} T' \xrightarrow{f'} S\}$ of f into an open immersion and a proper map is filtered. (Here, the morphisms in $\mathcal{C}$ are given by morphisms $T' \to T''$ respecting the factorization.)
(2) Show that the definition of $Rf_!$ is independent (up to isomorphism in $\mathcal{D}(S, \underline{\mathbb{Z}})$) of a relative compactification by using the proper base change theorem.
(3) Let $g\colon W \to T$ be another compactifiable morphism. Use the proper base change theorem to prove that there exists a natural isomorphism.

$$Rf_! \circ Rg_! \cong R(f \circ g)_!.$$

Let $j\colon W \to T$ be an open immersion. A priori we have two different functors $j_!, Rj_!\colon D(W, \underline{\mathbb{Z}}) \to D(T, \underline{\mathbb{Z}})$. The derived functor of the exact functor $j_!$ from Definition 4.8 and the functor $Rj_!$ from Definition 4.22. But as we can factor j into $W \xrightarrow{j} T \xrightarrow{\mathrm{Id}} T$, we see that $j_! = Rj_!$.

Let us give several remarks concerning the definition of $Rf_!$.

[25] Question: Is any separated map of locally compact spaces compactifiable in this sense, say by some relative one-point or Stone-Čech compactification?

Remark 4.24. (1) For a map $f\colon T \to S$ of locally compact Hausdorff spaces the functor $Rf_!$ can be constructed as the right derived functor of the functor

$$f_!\colon \mathrm{Sh}_{\mathrm{Ab}}(T) \to \mathrm{Sh}_{\mathrm{Ab}}(S)$$

sending $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T)$ to the sheaf $V \subseteq S \mapsto \{s \in \mathcal{F}(f^{-1}(V)) \mid \mathrm{supp}(s) \text{ is proper over } V\}$, which generalizes the functor $j_!$ from Definition 4.8. We did not take this approach as it will not generalize to étale cohomology. More details on $f_!$ can be found in [18, Chapter III].

(2) For an open immersion $j\colon W \to T$ the (exact) functor $j_!$ commutes with (all) colimits as it is left adjoint to the functor j^*. From the proper base change theorem and Lemma 4.20 we can conclude that for a proper map $f\colon T' \to T$ the functor $Rf_*\colon \mathcal{D}^{\geq 0}(T', \underline{\mathbb{Z}}) \to \mathcal{D}^{\geq 0}(T, \underline{\mathbb{Z}})$ commutes with filtered colimits of complexes. Indeed, by Theorem 4.16 we reduces to checking this a stalks, where it is implied by Lemma 4.20. We can deduce that the functor

$$Rf_!\colon \mathcal{D}^{\geq 0}(T, \underline{\mathbb{Z}}) \to \mathcal{D}^{\geq 0}(S, \underline{\mathbb{Z}})$$

commutes with all filtered colimits of complexes.

(3) Compactly supported cohomology is contravariantly functorial for *proper* maps $f\colon T \to S$ by applying $R\Gamma_c(S, -)$ to the unit $\mathcal{F} \to Rf_*(f^*\mathcal{F})$ for $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(S)$ and using $R\Gamma_c(S, Rf_*(-)) \cong R\Gamma_c(T, -)$ by Exercise 4.23.

(4) Compactly supported cohomology satisfies a certain *covariant* functoriality for open immersion $j\colon W \to T$. Indeed, for $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T)$ apply $R\Gamma_c(T, -)$ to the counit $j_!j^*\mathcal{F} \to \mathcal{F}$ and use $R\Gamma_c(T, j_!(-)) \cong R\Gamma_c(W, -)$, cf. Exercise 4.23). This yields a map

$$H^*_c(W, j^*\mathcal{F}) \cong H^*_c(T, \mathcal{F}).$$

(5) Note that the map $f\colon T \to \{*\}$ is compactifiable if and only if T is locally compact Hausdorff. This explains maybe a bit why usually sheaf cohomology is discussed in more detail only under this assumpation.

Compactly supported cohomology has very good computational purposes. For example, we get a Mayer-Vietoris sequence similar to the usual Mayer-Vietoris sequence

$$\ldots \to H^n(T, \mathcal{F}) \to H^n(U, \mathcal{F}_{|U}) \oplus H^n(V, \mathcal{F}_{|V}) \to H^n(U \cap V, \mathcal{F}_{|U\cap V}) \to \ldots$$

for a topological space T with an open cover $T = U \cup V$ (this long exact sequence can be deduced from the distinguished triangle $\mathcal{F} \to Rj_{U,*}(\mathcal{F}_{|U}) \oplus Rj_{V,*}(\mathcal{F}_{|V}) \to Rj_{U\cap V,*}(\mathcal{F}_{|U\cap V})$ in $\mathcal{D}(T, \mathbb{Z})$ as $R\Gamma(T, Rj_{U,*}(-) \cong R\Gamma(U, -)$ and similarly for V.)

Exercise 4.25. Let T be a locally compact Hausdorff space, $T = U \cup V$ an open cover and $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T)$. Then there exists a natural long exact sequence

$$\ldots \to H^n_c(U \cap V, \mathcal{F}_{|U\cap V}) \to H^n_c(U, \mathcal{F}_{|U}) \oplus H^n_c(V, \mathcal{F}_{|V}) \to H^n_c(T, \mathcal{F}_{|T}) \to \ldots.$$

Using $Rf_!$ the proper base change theorem can now be generalized.

Theorem 4.26. *Consider a cartesian diagram*

$$\begin{array}{ccc} S' & \xrightarrow{g'} & T' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{g} & T \end{array}$$

of locally compact spaces with f compactifiable, then there exists a natural isomorphism

$$g^*Rf_! \cong Rf'_!g'^{,*}$$

of functors $\mathcal{D}^+(T', \underline{\mathbb{Z}}) \to \mathcal{D}^+(S, \underline{\mathbb{Z}})$.

If $g\colon S = \{t\} \to T$ is the inclusion of the point $t \in T$, then Theorem 4.26 implies

$$(Rf_!(K))_t \cong R\Gamma_c(f^{-1}(t), K_{|f^{-1}(t)}),$$

which is a very useful fact.

Proof. This follows from Remark 4.17 and the easy case that $f = j$ is an open immersion. □

We have established the proper base change in order to obtain (an important case of) excision. But the proper base change theorem (by which we mean either Theorem 4.16 or Theorem 4.26) in itself is very useful for computations.

4.27. **Some explicit examples.** Let us compute some concrete examples of Betti cohomology via the proper base change theorem.

Example 4.28. Let $X = \mathrm{Spec}(\mathbb{C}[x,y]/(y^2-x^3))$ be the cuspidal curve with normalization $f\colon Y \to X$. Then $Y \cong \mathbb{A}^1_{\mathbb{C}}$. Then $Y^{\mathrm{an}} \to X^{\mathrm{an}}$ is a homeomorphism and we can conclude that

$$H^*_{\mathrm{Betti}}(X,\mathbb{Z}) \cong H^*(\mathbb{C},\mathbb{Z}).$$

More generally, if $f\colon Y \to X$ is any universal homeomorphism in $(\mathrm{Sch}^{\mathrm{loft}}_{/\mathbb{C}})$, then $H^*_{\mathrm{Betti}}(X,\mathbb{Z}) \cong H^*_{\mathrm{Betti}}(Y,\mathbb{Z})$.

Example 4.29. Let $X \subseteq \mathrm{Spec}(\mathbb{C}[x,y]/(y^2-x^3-x^2))$ be the compactified nodal curve with normalization $f\colon Y \cong \mathbb{P}^1_{\mathbb{C}} \to X$. Let $i\colon \{x_0\} \to X$ be the inclusion of the singular point. Then Theorem 4.16 implies that there exists a distinguished triangle

$$\mathbb{Z} \to Rf^{\mathrm{an}}_*(\mathbb{Z}) \to i_*\mathbb{Z}.$$

Indeed, the natural morphism $\mathbb{Z} \to Rf^{\mathrm{an}}_*(\mathbb{Z}) = Rf_*f^{\mathrm{an},*}(\mathbb{Z})$ is an isomorphism over $X^{\mathrm{an}} \setminus \{x_0\}$ and by Theorem 4.16 we can calculate that $\mathbb{Z}_{x_0} \to Rf^{\mathrm{an}}_*(\mathbb{Z})_{x_0} \cong R\Gamma(f^{\mathrm{an},-1}(x_0),\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$ is the diagonal inclusion. In particular, we see that $Rf^{\mathrm{an}}_*(\mathbb{Z}) \cong f_*(\mathbb{Z})$ (more generally, if $f\colon Y \to X$ is any *finite* morphism between schemes, locally of finite over $\mathbb{C}$, then the functor $(f^{\mathrm{an}})_*$ is exact). Hence, there exists a distinguished triangle

$$R\Gamma(X^{\mathrm{an}},\mathbb{Z}) \to \underbrace{R\Gamma(\mathbb{C}P^1,\mathbb{Z})}_{\cong R\Gamma(X^{\mathrm{an}},Rf^{\mathrm{an}}_*(\mathbb{Z}))} \to \underbrace{R\Gamma(X^{\mathrm{an}},i_*\mathbb{Z})}_{\cong\mathbb{Z}[0]}$$

and we can conclude that

$$H^i(X^{\mathrm{an}},\mathbb{Z}) = \begin{cases}\mathbb{Z}, & i = 0,1,2\\ 0, & i > 2.\end{cases}$$

Example 4.30. Let $X = \mathrm{Spec}(\mathbb{C}[x,y,z]/(z^2-xy))$ be the cone and $f\colon Y \to X$ be the blow-up at the point $x_0 := (0,0,0)$. Let $i\colon \{x_0\} \to X$ be the inclusion. Then $f^{-1}(x_0) \cong \mathbb{P}^1_{\mathbb{C}}$ and $Y \cong \mathbb{V}(\mathcal{O}(-2))$ is the total space of the line bundle $\mathcal{O}(-2)$ on $\mathbb{P}^1_{\mathbb{C}}$. In particular, $R\Gamma(Y^{\mathrm{an}},\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}[-2]$ by homotopy invariance. Using the proper base change theorem we get a distinguished triangle

$$\mathbb{Z} \to Rf_*(\mathbb{Z}) \to i_*(\mathbb{Z}[-2])$$

and from here a distinguished triangle

$$R\Gamma(X^{\mathrm{an}},\mathbb{Z}) \to \mathbb{Z}\oplus\mathbb{Z}[-2] \to \mathbb{Z}[-2].$$

Now, the occuring morphism $\mathbb{Z}[-2] \to \mathbb{Z}[-2]$ is the identity (because $f^{-1}(x_0) \subseteq Y$ is the zero-section of the line bundle $\mathbb{V}(\mathcal{O}(-2))$). Hence, $R\Gamma(X^{\mathrm{an}},\mathbb{Z}) \cong \mathbb{Z}$, which is good because drawing a picture we'd actually want that X^{an} is contractible. Using the scaling multiplication $\mathbb{A}^1_{\mathbb{C}} \times X \to X,\ (t,(x,y,z)) \mapsto (tx,ty,tz)$ we can actually write down a contradiction. This example generalizes to the vanishing locus of some set of homogeneous polynomial in some $\mathbb{A}^n_{\mathbb{C}}$.

Exercise 4.31. Let $X \to \mathrm{Spec}(\mathbb{C})$ be a morphism locally of finite type and let $Z \subseteq X$ be a closed subscheme. Let $f\colon Y \to X$ be the blow-up of X in Z and let $E \subseteq Y$ be the exceptional divisor. Show the existence of a natural long exact sequence

$$\to \ldots \to H^n(X^{\mathrm{an}},\underline{\mathbb{Z}}) \to H^n(Y^{\mathrm{an}},\underline{\mathbb{Z}}) \oplus H^n(Z^{\mathrm{an}},\underline{\mathbb{Z}}) \to H^n(E^{\mathrm{an}},\underline{\mathbb{Z}}) \to \ldots.$$

Hint: Use proper base change and make an analysis as in Example 4.30 for $Rf^{\mathrm{an}}_*(\mathbb{Z}) \oplus i^{\mathrm{an}}_{Z,*}(\mathbb{Z})$*, where* $i_Z\colon Z \to X$ *is the inclusion.*

4.32. **Dimension of locally compact Hausdorff spaces.** In this section we want to prove some results on the cohomological dimension of real manifolds.

Let T be a locally compact Hausdorff space.

Definition 4.33. The (cohomological) dimension $\dim(T)$ of T is the infimum of all integers d, such that

$$H^i_c(T,\mathcal{F}) = 0$$

for all $i > d$ and all sheaves $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T)$.

The dimension satisfies some useful stabilities.

Lemma 4.34. . *Let* $f\colon T' \to T$ *be a morphism of locally compact Hausdorff spaces.*

(1) *If* f *is a locally closed immersion, then* $\dim(T') \leq \dim(T)$.
(2) *If there exists some* $d \geq 0$*, such that* $\dim(f^{-1}(t)) \leq d$ *for all* $t \in T$*, then* $\dim(T') \leq \dim(T) + d$.

Proof. If $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T')$, then $Rf_! \cong f_!$ as f is a locally closed immersion. This implies that

$$R\Gamma_c(T', \mathcal{F}) = R\Gamma_c(T, f_!(\mathcal{F})) \in D^{\leq \dim(T)}(\mathbb{Z})$$

as desired. In the second case note that $Rf_!(\mathcal{F}) \in D^{\leq d}(T, \underline{\mathbb{Z}})$ by Theorem 4.26 and the assumption. Thus,

$$R\Gamma_c(T', \mathcal{F}) = R\Gamma_c(T, Rf_!(\mathcal{F})) \in \mathcal{D}^{\leq d + \dim(T)}(\mathbb{Z})$$

as desired. □

Theorem 4.35. *Let T be a (Hausdorff) real manifold of dimension d. Then the cohomological dimension of T is d. In particular, each locally closed subset of a (Hausdorff) real manifold of dimension d is of cohomological dimension $\leq d$*

Before proving the theorem let us establish a lemma.

Lemma 4.36. *Let T be a locally compact Hausdorff space and $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T)$. Let $\alpha \in H^k_c(T, \mathcal{F})$ be non-zero for some $k \geq 0$. Then there exists a closed set $Z \subseteq T$ such that $\alpha_{|Z} \neq 0$, but $\alpha_{|A} = 0$ for all closed subsets $A \subsetneq Z$.*

Proof. Consider the partially ordered set J of all $Z \subseteq T$ closed with $\alpha_{|Z} \neq 0$, where $Z \leq Z'$ if $Z' \subseteq Z$. Then J is non-empty and for each filtered system $Z_i, i \in I$, of elements in J also $Z := \bigcap\limits_{i \in I} Z_i \in J$. Indeed, if $i_{Z_i} \colon Z_i \to T$ denotes the closed immersion, then

$$\varinjlim_{i \in I} i_{Z_i,*} i^*_{Z_i} \mathcal{F} \cong i_{Z,*} i^*_Z \mathcal{F}$$

by looking at stalks, and thus

$$\varinjlim_{i \in I} \underbrace{H^k_c(T, i_{Z_i,*} i^*_{Z_i} \mathcal{F})}_{\cong H^k_c(Z_i, \mathcal{F}_{|Z_i})} \cong H^k_c(Z, \mathcal{F}_{|Z})$$

by Remark 4.24. This implies the claim. Thus, we can apply Zorn's lemma to J and conclude. □

Proof of Theorem 4.35. Let us check that $\dim(T) \leq d$. Let $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T)$. First assume that $T = \mathbb{R}$ and let $\omega \in H^k_c(T, \mathcal{F})$ be non-zero, $k \geq 0$. Let $Z \subseteq T$ be a minimal closed set for ω as in Lemma 4.36. If $Z = \{z\}$ is a point, then necessarily $k = 0$ as $H^k_c(\{z\}, \mathcal{F}) \cong H^k(\{z\}, \mathcal{F}) \neq 0$. If Z is not a point, then there exists some $c \in Z$, such that $Z^- := (-\infty, c] \cap Z, Z^+ := [c, \infty) \cap Z$ are proper closed subsets of Z, and hence by assumption on Z we have $\omega_{|Z^+} = 0, \omega_{|Z^-} = 0$. Now, we have a long exact sequence

$$\to \ldots \to H^{k-1}_c(\{c\}, \mathcal{F}) \to H^k_c(Z, \mathcal{F}) \to H^k_c(Z^+, \mathcal{F}) \oplus H^k_c(Z^-, \mathcal{F}) \to$$

(induced from the short exact sequence $0 \to \mathcal{F} \to i_{Z^+,*} i^*_{Z^+} \mathcal{F} \oplus i_{Z^-,*} i^*_{Z^-} \mathcal{F} \to i_{\{c\},*} i^*_{\{c\}} \mathcal{F} \to 0$) and thus the element $\omega \in H^k_c(Z, \mathcal{F})$ lifts to a non-zero element in $H^{k-1}_c(\{z\}, \mathcal{F})$. Hence, $k-1 = 0$, i.e., $k = 1$. This finishes the proof in the case $T = \mathbb{R}$. By Lemma 4.34 we can deduce $\dim(T) \leq d$ for any locally closed subset T of $\mathbb{R}^n$. Assume that T is a general (Hausdorff) real manifold. Then we can write $\mathcal{F}$ as a colimit of sheaves $j_{U,!} j^*_U \mathcal{F}$ for $U \subseteq T$ open, and covered by finitely many open balls in $\mathbb{R}^d$. Hence, we may (as $H^*_c(T, -)$ commutes with filtered colimits) assume that T is covered by finitely open subsets of $\mathbb{R}^n$. By Exercise 4.25 and induction on the number of open subsets we can conclude. □

We leave it as an exercise to check that the cohomological dimension of $\mathbb{R}^d$ is exactly d.

4.37. **Homotopy invariance of sheaf cohomology.** Curiously, all our computations relied up to now on Theorem 2.16, e.g., via the use of homotopy invariance for example. In this section we want to remedy this.

Let T be any topological space and $\mathcal{G}$ a sheaf of (not necessarily abelian) groups on T. Let us recall the definition of a G-torsor, or G-principal homogeneous space.

Definition 4.38. A $\mathcal{G}$-torsor on T is a sheaf of set $\mathcal{P}$ with a (right) action $\mathcal{P} \times \mathcal{G} \to \mathcal{P}$ of $\mathcal{G}$, such that there exists an open cover $T = \bigcup\limits_{i \in I} U_i$, such that $\mathcal{P}_{|U_i} \cong \mathcal{G}_{|U_i}$ as sheaves with $\mathcal{G}_{|U_i}$-action.

Torsors yield a concrete interpretation of the first sheaf cohomology.

Lemma 4.39. *Let T be a topological space and $\mathcal{G}$ a sheaf of abelian groups on T. Then there exists a natural (in $T, \mathcal{G}$) isomorphism*

$$H^1(T, \mathcal{G}) \cong \{\mathcal{G} - \textit{torsors}\}/\textit{isom}.$$

Proof. Cf. [Stacks, Tag 02FN]. □

Now assume that $\mathcal{G} = \underline{G}$ is a constant sheaf for some abelian group G and let $\mathcal{P}$ be a G-torsor with trivializations $\alpha_i \colon \mathcal{P}_{|U_i} \cong \mathcal{G}_{|U_i}$ on some open cover $T = \bigcup_{i\in I} U_i$. Then for $i,j \in I$ the isomorphism

$$\alpha_i \circ \alpha_j^{-1} \colon \mathcal{G}_{|U_i\cap U_j} \cong \mathcal{G}_{|U_i\cap U_j}$$

must be given by left multiplication by some section $g_{ij} \colon U_i \cap U_j \to G$ (as it is equivariant for the $\mathcal{G}_{|U_i\cap U_j}$-action). Now, we glue the topological spaces $U_i \times G = \coprod_{g\in G} U_i \times \{g\}$ along the isomorphism

$$(U_i \cap U_j) \times G \to (U_i \cap U_j) \times G,\ (u,g) \mapsto (u, g_{ij}(u)g)$$

because the $g_{i,j}$ satisfy the cocycle condition. This glueing yields a topological covering space $P \to X$, and the construction $\mathcal{P} \mapsto P$ is a fully faithful functor from $\underline{G}$-torsors to topological spaces over X, which are equipped with a G-action over X.

In fact, we leave the following statement as an exercise.

Exercise 4.40. For a morphism $Y \to T$ let $\underline{Y}_T$ be the sheaf $U \mapsto \mathrm{Hom}_T(U,Y)$. Show that the functor

$$\{\text{local isomorphisms } Y \to T\} \to \mathrm{Sh}(T),\ Y \mapsto \underline{Y}_T$$

is an equivalence (the morphisms on the left hand side are morphisms over T), and its inverse maps $\mathcal{P}$ to P in the above situation.

We get the following corollary.

Lemma 4.41. *Let T be a simply connected topological space and G an abelian group. Then*[26]

$$H^1(T,G) = \{*\}.$$

In particular, $H^i([0,1],G) \cong G$ if $i=0$ and 0 if $i>0$.

Proof. In the above notation the topological covering space $P \to T$ must have a section $T \to P$. Thus, we can conclude by Exercise 4.40 (or just the existence of an isomorphism $\underline{P}_T \cong \mathcal{P}$) that $\sigma\mathcal{P}(T) \neq \emptyset$. But then the map $\underline{G} \to \mathcal{P},\ g \mapsto \sigma \cdot g$ is an isomorphism of $\underline{G}$-torsors. The statement for $[0,1]$ follows from Theorem 4.35. □

Now, we leave the following generalization of homotopy invariance as an exercise in using Lemma 4.41 and Theorem 4.16.[27]

Exercise 4.42. . Let T be a locally compact space and $f \colon T \times [0,1] \to T$ the projection. Then for any $K \in \mathcal{D}^+(T,\underline{\mathbb{Z}})$ the natural map

$$K \to Rf_*f^*K$$

is an isomorphism.

At this point we are not dependent on any result on singular cohomology anymore. For example, we can prove finiteness of cohomology.

Exercise 4.43. Let T be a finite CW-complex. Show that for each $i \geq 0$ the group

$$H^i(T,\underline{\mathbb{Z}})$$

is finitely generated. *Hint: Use excision Section 4.7 for a closed subcomplex A.*

We have developed enough results on sheaf cohomology for the moment and it is time to turn to developing étale cohomology of schemes.

[26] In general, $H^1(T,G) \cong \mathrm{Hom}(\pi_1(T,t),G)$ if T is connected and, locally simply connected and $\pi_1(T,t)$ its first fundamental group for some $t \in T$ as follows by some easy arguments using Exercise 4.40 and the fact that representations of $\pi_1(T,t)$ classify all covering spaces of T.

[27] We note that at this point we have established all Eilenberg-Steenrod axioms for sheaf cohomology with coefficients in $\underline{\mathbb{Z}}$, at least on (locally compact, Hausdorff) CW-complexes. Amusingly, the homotopy variance was the last one that we considered while for singular cohomology it is usually the first statement to be proven.

5. Topoi and étale cohomology

Our aims for the rest of this course are the following:

(1) the definition of étale cohomology for schemes,
(2) some general results on étale cohomology (or more generally, cohomology of topoi),
(3) the cohomology of curves over algebraically closed fields,
(4) the proper base change theorem in étale cohomology.

5.1. **Étale sheaves.** In the next subsections we will develope the necessary abstract sheaf theory to finally reach the definition of étale cohomology of an arbitrary scheme (without being able to calculate anything with it for some time).

Definition 5.2. Let X be a scheme.

(1) We define the category
$$X_{\text{ét}} := \{Y \to X \text{ étale}\}$$
with morphisms given by morphisms over X. Note that each morphism in $X_{\text{ét}}$ is étale, cf. Lemma 10.26.
(2) An étale presheaf (of sets) is a functor $\mathcal{F}\colon X_{\text{ét}}^{\text{op}} \to (\text{Sets})$ and a morphism between them is a natural transformation. For $f\colon Y \to Z$ in $X_{\text{ét}}$ we denote by $f^*\colon \mathcal{F}(Z) \to \mathcal{F}(Y)$ (or $s \mapsto s_{|Y}$ if f is clear) the restriction morphism for $\mathcal{F}$.
(3) An étale presheaf $\mathcal{F}$ is an étale sheaf if for all $Y \in X_{\text{ét}}$ and all collections of étale morphisms $f_i\colon Y_i \to Y, i \in I$, such that the the f_i are jointly surjective, i.e., $Y = \bigcup\limits_{i\in I} f_i(Y_i)$, the sequence

$$\mathcal{F}(Y) \to \prod_{i\in I} \mathcal{F}(Y_i) \rightrightarrows \prod_{i,j} \mathcal{F}(Y_i \times_Y Y_j) \tag{1}$$

is exact. Here, the first arrow sends $s \in \mathcal{F}(Y)$ to $(f_i^*(s))_i$ and the two other arrows are given by
$$(s_i)_{i\in I} \mapsto ((p_1^*(s_i))_{i,j},\ (s_i)_{i\in I} \mapsto ((p_2^*(s_j))_{i,j},$$
where $p_1\colon Y_i \times_Y Y_j \to Y_i,\ p_2\colon Y_i \times_Y Y_j \to Y_j$ are the two projections.
(4) A morphism of étale sheaves is a natural transformation.
(5) We denote by $\text{PSh}(X_{\text{ét}})$ the category of étale presheaves, and by $\text{Sh}(X_{\text{ét}})$ or $\widetilde{X_{\text{ét}}}$ its full subcategory of étale sheaves.

If in (Equation (1)) the map $\mathcal{F}(Y) \to \prod\limits_{i\in I} \mathcal{F}(Y_i)$ is always injective, then we call $\mathcal{F}$ a separated étale presheaf.

We denote by X_{Zar} the full subcategory of $X_{\text{ét}}$ given by open immersions $Y \to X$. Note that when restricting the condition of an étale sheaf to this subcategory, we exactly get the condition of being a sheaf on the topological space $|X|$.

Remark 5.3. The definition of $X_{\text{ét}}$ and of an étale sheaf can be motivated by contemplating Exercise 4.40. Namely, the category $\text{Sh}(X_{\text{Zar}}) = \widetilde{X_{\text{Zar}}}$ of sheaves on $|X|$ is equivalent to the category $\{Y \to X \text{ local isomorphism }\}$. Contrary to complex analytic spaces the notions “étale” and “local isomorphism” differ for schemes, and thus using étale morphisms we get something different.

Exercise 5.4. Let X be a complex analytic space and define étale sheaves on it exactly as in Definition 5.2. Show that $\text{Sh}(X_{\text{ét}})$ is equivalent to the usual category of sheaves on $|X|$.

Example 5.5. From the pointwise construction of limits in $\text{PSh}(X_{\text{ét}})$ and by commuting limits with products it follows that the full subcategory $\text{Sh}(X_{\text{ét}}) \subseteq \text{PSh}(X_{\text{ét}})$ is stable under limits. In particular, $\text{Sh}(X_{\text{ét}})$ admits all limits.

Let us now establish the sheafification of étale presheaves. First we introduce the following convenient terminology.

Definition 5.6. Let X be a scheme, and $Y \in X_{\text{ét}}$.

(1) An (étale) covering $\mathcal{U}$ of Y is a collection $\{f_i\colon Y_i \to Y\}_{i\in I}$ of étale morphisms to Y, which is jointly surjective, i.e., $Y = \bigcup\limits_{i\in I} f_i(Y_i)$.
(2) A covering $\mathcal{U} = \{Y_i \to Y\}_{i\in I}$ refines a covering $\mathcal{V} = \{Z_j \to Y\}_{j\in J}$ if there exists a map $\varphi\colon I \to J$ and morphisms $g_i\colon Y_i \to Z_{\varphi(i)}, i \in I$, over Y.

(3) Given a covering $\mathcal{U} = \{f_i\colon Y_i \to Y\}_{i\in I}$ and $\mathcal{F} \in \mathrm{PSh}(X_{\text{ét}})$ we set

$$\Gamma(\mathcal{U},\mathcal{F}) := \mathrm{eq}(\prod_{i\in I}\mathcal{F}(Y_i) \rightrightarrows \prod_{i,j}\mathcal{F}(Y_i\times_Y Y_j)),$$

where the morphisms are defined as in Definition 5.2.

The following arguments will be quite formal and only use the following properties of étale coverings:

(Isomorphisms are covers) If $Z \to Y$ is an isomorphism, then $\{Z \to Y\}_{I=\{*\}}$ is a covering.

(Covers can be composed) If $\{Y_i \to Y\}_{i\in I}$ is a covering and $\{Z_{i,j} \to Y_i\}_{j\in J_i}$ is a covering for each $i \in I$, then $\{Z_{i,j} \to Y\}_{i\in I, j\in J_i}$ is a covering.

stable under base change) If $\{Y_i \to Y\}_{i\in I}$ and $Z \to Y$ is a morphism, then $Y_i \times_Y Z$ exists for any $i \in I$ and $\{Y_i \times_Y Z \to Z\}_{i\in I}$ is a covering.

(Set-theoretic smallness) For each $Y \in X_{\text{ét}}$ there exists a set of coverings $\{\mathcal{U}\}$ such that each covering of Y can be refined by on these $\mathcal{U}$'s, cf. Remark 5.9.

In Definition 5.11 we will abstract the first three conditions into the definition of a site.

Lemma 5.7. *Assume that the covering $\mathcal{U} = \{Y_i \to Y\}_{i\in I}$ refines the cover $\mathcal{V} = \{Z_j \to Y\}_{j\in J}$. Assume that $\varphi\colon I \to J$ is a map, and $g_i\colon Y_i \to Z_{\varphi(i)}, i \in I$, morphisms over Y. Let $\mathcal{F} \in \mathrm{PSh}(X_{\text{ét}})$.*

(1) *The map*

$$r_{\mathcal{U},\mathcal{V}}\colon \Gamma(\mathcal{V},\mathcal{F}) \to \Gamma(\mathcal{U},\mathcal{F}),\ (s_j)_{j\in J} \mapsto (g_i^*(s_{\varphi(i)}))_{i\in I}$$

is well defined and independent of the choice of $\varphi, g_i, i \in I$.

(2) *If $\mathcal{F}$ is separated and the natural map $\mathcal{F}(Y) \to \Gamma(\mathcal{U},\mathcal{F})$ is bijective, the same holds for $\mathcal{F}(Y) \to \Gamma(\mathcal{V},\mathcal{F})$.*

Proof. The well-definedness follows easily by using the map $(g_{i_1}, g_{i_2})\colon Y_{i_1}\times_Y Y_{i_2} \to Z_{\varphi(i_1)}\times_Y Z_{\varphi(i_2)}$.

Let now $j \in J$. Assume that $h_i\colon Y_i \to Z_j$ is any map over Y. Then we have a map $h := (g_i, h_i)\colon Y_i \to Z_{\varphi(i)} \times_Y Z_j$. Let $p_1\colon Z_{\varphi(i)} \times_Y Z_j \to Z_{\varphi(i)}$, $p_2\colon Z_{\varphi(i)} \times_Y Z_j \to Z_j$ be the projections. Then

$$g_i^*(s_{\varphi(i)}) = h^*(p_1^*(s_{(\varphi(i))})) \overset{(s_j)_j\in\Gamma(\mathcal{V},\mathcal{F})}{=} h^*(p_2^*(s_j)) = h_i^*(s_j).$$

This implies that $r_{\mathcal{U},\mathcal{V}}$ is independent of $\varphi, g_i, i \in I$.

Now, assume that the map $r_{\mathcal{U}}\colon \mathcal{F}(V) \to \Gamma(\mathcal{U},\mathcal{F})$ is bijective. Then clearly,

$$r_{\mathcal{V}}\colon \mathcal{F}(V) \to \Gamma(\mathcal{V},\mathcal{F})$$

is injective, as its composite with $r_{\mathcal{U},\mathcal{V}}$ is $r_{\mathcal{U}}$. Take now $(s_j)_j \in \Gamma(\mathcal{V},\mathcal{F})$. Then $r_{\mathcal{U},\mathcal{V}}((s_j)_j) = r_{\mathcal{U}}(s)$ for some $s \in \mathcal{F}(Y)$. Now fix $j \in J$ and consider $f_j\colon Z_j \to Y$. We need to see that $s_{|Z_j} = f_j^*(s) = s_j$. The collection

$$\{h_{i,j}\colon Y_i \times_Y Z_j \to Z_j\}_{i\in I}$$

is an (étale) covering of Z_j. Let $g_{i,j}\colon Y_i \times_Y Z_j \to Y_i$ be the projection. Then for all $i \in I$

$$h_{i,j}^*(f_j^*(s)) = g_{i,j}^*(s_{|Y_i}) = g_{i,j}^*(g_i^*(s_{\varphi(i)}))$$

and if $g := (g_i, \mathrm{Id}_{Z_j})\colon Y_i \times_Y Z_j \to Z_{\varphi(i)} \times_Y Z_j$ this equals

$$g^*(s_{\varphi(i))|Z_{\varphi(i)}\times_Y Z_j}) = g^*(s_{j|Z_{\varphi(i)}\times_Y Z_j}) = h_{i,j}^*(s_j)$$

by definition of $\Gamma(\mathcal{V},\mathcal{F})$. If now $\mathcal{F}$ is separated, this implies that

$$f_j^*(s) = s_j$$

and thus the second statement is proven. □

Definition 5.8. Let $\mathcal{F} \in \mathrm{PSh}(X_{\text{ét}})$, then we define the presheaf $\mathcal{F}^+ \in \mathrm{PSh}(X_{\text{ét}})$ as

$$(Y \in X_{\text{ét}}) \mapsto \mathcal{F}^+(Y) := \varinjlim_{\mathcal{U}\text{ covering of }Y} \Gamma(\mathcal{U},\mathcal{F}),$$

where the colimit is taken along the maps $r_{\mathcal{U},\mathcal{V}}$ from Lemma 5.7 if $\mathcal{U}$ refines $\mathcal{V}$. If $f\colon Y \to Z$ is a morphism in $X_{\text{ét}}$ and $\mathcal{U} = \{Z_j \to Z\}_{j\in J}$ a covering of Z, then $\mathcal{U}\times_Z Y := \{Y \times_Z Z_j \to Z_j\}_{j\in J}$ is a covering of Y and the pullbacks along $Y \times_Z Z_j \to Z_j$ define a natural morphism

$$\Gamma(\mathcal{U},\mathcal{F}) \to \Gamma(\mathcal{U}\times_Z Y,\mathcal{F}),$$

which is compatible with refinement. Passing to the colimit over all coverings yields the restriction morphism

$$\mathcal{F}^+(Z) \to \mathcal{F}^+(Y)$$

for $\mathcal{F}^+$.

Remark 5.9. (1) The colimit $\varinjlim_{\mathcal{U} \text{ covering of } Y}$ is *filtered.* Indeed, given two coverings $\{Y_i \to Y\}_{i\in I}$, $\{Z_j \to Z\}_{j\in J}$ are two coverings, then $\{Y_i \times_Y Z_j \to Y\}_{(i,j)\in I\times J}$ is a common refinement of both.

(2) We have to make the following set-theoretic warning that a priori the colimit over all coverings $\mathcal{U}$ might be over some *class.* But we can find a cofinal *set* of coverings as follows: First of all each covering $\{Y_i \to Y\}_{i\in I}$ admits a refinement by a cover for which $|I| \leq |Y|$. Refinining further we may assume that Y_i is affine, with image contained in some affine in Y. But for $Y = \mathrm{Spec}(A)$ affine there exists a *set* of isomorphism classes of A-algebras B, which are of finite presentation. As étale morphisms are locally of finite presentation this yields the desired cofinal *set* of coverings.

We can now establish the sheafification of étale presheaves.

Theorem 5.10. (1) *If $\mathcal{F} \in \mathrm{PSh}(X_{\text{ét}})$, then the presheaf $\mathcal{F}^+$ is separated.*

(2) *If $\mathcal{F} \in \mathrm{PSh}(X_{\text{ét}})$ is separated, then $\mathcal{F}^+$ is an étale sheaf and $\mathcal{F} \to \mathcal{F}^+$ is the initial morphism from $\mathcal{F}$ to an étale sheaf.*

In particular, the functor $\mathcal{F} \mapsto \mathcal{F}^\sharp := (\mathcal{F}^+)^+$ yields a left adjoint ("sheafification") to the inclusion $\mathrm{Sh}(X_{\text{ét}}) \to \mathrm{PSh}(X_{\text{ét}})$. Finally, the sheafification $(-)^\sharp\colon \mathrm{PSh}(X_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$ is an exact functor.

Proof. The first statement follows easily from the definition of $\mathcal{F}^+$ and of being a separated presheaf. Let us prove the second and assume that $\mathcal{F}$ is a separated presheaf. We have to show that $\mathcal{F}^+$ is an étale sheaf. Thus fix $Y \in X_{\text{ét}}$ and a covering $\mathcal{U} = \{Y_i \to Y\}_{i\in I}$. We have to show that

$$\alpha\colon \mathcal{F}^+(Y) \to \Gamma(\mathcal{U}, \mathcal{F}^+)$$

is bijective. As $\mathcal{F}^+$ is separated (by (1)) the map α is injective. Let

$$(s_i)_{i\in I} \in \Gamma(\mathcal{U}, \mathcal{F}^+).$$

Then there exist coverings $\mathcal{U}_i = \{Z_{i,j} \to Y_i\}_{j\in J_i}$ such that $s_i \in \mathcal{F}^+(Y_i)$ can be represented by some $(t_{i,j})_{j\in J_i} \in \Gamma(\mathcal{U}_i, \mathcal{F})$. Now the collection of morphisms

$$\mathcal{V} = \{Z_{i,j} \to Y_i \to Y\}_{i\in I, j\in J_i}$$

is a covering of Y. By construction, the element

$$r_{\mathcal{V},\mathcal{U}}((s_i)_i) \in \Gamma(\mathcal{V}, \mathcal{F}^+)$$

lies in the subset $\Gamma(\mathcal{V}, \mathcal{F}^+) \cap \prod_{i,j} \mathcal{F}(Z_{i,j})$ (here we identify $\mathcal{F}(Z_{i,j}) \subseteq \mathcal{F}^+(Z_{i,j})$ as a subset by separatedness of $\mathcal{F}$). Now, if $\mathcal{G}$ is any separated presheaf and $\mathcal{W} = \{Z_k \to Z\}_{k\in K}$ one has the equality

$$\Gamma(\mathcal{W}, \mathcal{G}^+) \cap \prod_{k\in K} \mathcal{G}(Z_i) = \Gamma(\mathcal{W}, \mathcal{G})$$

because the map $\prod_{k,l\in K} \mathcal{G}(Z_k \times_Z Z_l) \to \prod_{k,l\in K} \mathcal{G}^+(Z_k \times_Z Z_l)$ is injective. Applied in our situation we see that

$$t := r_{\mathcal{U},\mathcal{V}}((s_i)_i) \in \Gamma(\mathcal{V}, \mathcal{F}) = \Gamma(\mathcal{V}, \mathcal{F}^+) \cap \prod_{i,j} \mathcal{F}(Z_{i,j}).$$

But this implies that $t \in \Gamma(\mathcal{V}, \mathcal{F}) \subseteq \mathcal{F}^+(Y)$ restricts to $(s_i)_i \in \Gamma(\mathcal{V}, \mathcal{F}^+)$ as desired. It follows from the definition that a presheaf $\mathcal{F}$ is sheaf if and only if that map $\mathcal{F} \to \mathcal{F}^\sharp$. This implies that $(-)^\sharp$ satisfies the universal property of sheafification. In particular, it is left adjoint to the inclusion $\mathrm{Sh}(X_{\text{ét}}) \subseteq \mathrm{PSh}(X_{\text{ét}})$. It suffices to see that $(-)^\sharp$ commutes with finite limits. By Example 5.5 limits in $\mathrm{Sh}(X_{\text{ét}})$ agree with limits in $\mathrm{PSh}(X_{\text{ét}})$. As filtered colimits commute with finite limits, it follows by Remark 5.9 that the functor $(-)^+\colon \mathrm{PSh}(X_{\text{ét}}) \to \mathrm{PSh}(X_{\text{ét}})$ commutes with finite limits. In particular, $(-)^\sharp = ((-)^+)^+$ commutes with finite limits. □

Thus, the categories of étale (pre)sheaves on $X_{\text{ét}}$ satisfies the same formal properties as the categories of usual (pre)sheaves. For example, $\mathrm{Sh}(X_{\text{ét}})$ admits all colimits and these can be calculated by sheafifying the colimit in $\mathrm{PSh}(X_{\text{ét}})$.

The necessary properties for étale coverings can be abstracted in the definition of a site.

Definition 5.11. (1) If $\mathcal{C}$ is a category, we define the category $\mathrm{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$ as the category of functors $\mathcal{C}^{\mathrm{op}} \to (\mathrm{Sets})$.

(2) A site is a category $\mathcal{C}$ together with a collection τ of collections of morphisms $\{Y_i \to Y\}_{i\in I}$ for each $Y \in \mathcal{C}$ (called coverings in τ, or just coverings) such that the above properties (Isomorphisms are covers), (Covers can be composed) and (Covers are stable under base change) are satisfied.[28]
(3) A presheaf $\mathcal{F} \in \mathrm{PSh}(\mathcal{C})$ is called a sheaf (for τ) if for any covering $\{Y_i \to Y\}_{i\in I}$ in τ the sequence

$$\mathcal{F}(Y) \to \prod_{i\in I} \mathcal{F}(Y_i) \rightrightarrows \prod_{i,j} \mathcal{F}(Y_i \times_Y Y_j)$$

is exact. A morphism of sheaves is a natural transformation of functors.
(4) We denote by $\mathrm{Sh}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ the full subcategory of sheaves (with respect to τ) on the site $\mathcal{C}$.

As in the case of usual sheaves the category $\mathrm{Sh}(\mathcal{C})$ has all limits and these can be calculated in $\mathrm{PSh}(\mathcal{C})$. By design for a scheme X the category $X_{\text{ét}}$ equipped the class of coverings as in Definition 5.2 is a site, called the étale site of X.

Example 5.12. Let $\mathcal{C}$ be any category.

(1) The indiscrete Grothendieck topology on $\mathcal{C}$ is given by the covers $\{f\colon Z \to Y\}$ with f an isomorphism. In this case,

$$\mathrm{Sh}(\mathcal{C}) = \mathrm{PSh}(\mathcal{C})$$

is just the category of functors $\mathcal{C}^{\mathrm{op}} \to (\mathrm{Sets})$.
(2) Assume that $\mathcal{C}$ has fiber products. Then we can define the Grothendieck topology, whose coverings are just any collections $\{Y_i \to Y\}_{i\in I}$. The $\mathrm{Sh}(\mathcal{C}) \cong \mathrm{Sh}(\pi_0(\mathcal{C})) = \mathrm{PSh}(\pi_0(\mathcal{C}))$, where $\pi_0(\mathcal{C})$ denotes the set of isomorphism classes in $\mathcal{C}$ (assuming that this is indeed a set).

Theorem 5.13. *Let $\mathcal{C}$ be a site, and let τ be its Grothendieck. If τ satisfies the above property (Set-theoretic smallness), then the inclusion $\mathrm{Sh}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ admits an exact, left adjoint $(-)^\sharp$.*

Proof. The proof of Theorem 5.10 applies literally in the same way. ☐

In particular, we can as usual consider sheaves of abelian groups, sheaves of rings, sheaves of Λ-modules for some ring Λ, $\mathcal{G}$-torsors for a sheaf of groups $\mathcal{G}$, free abelian sheaf associated to some sheaf,... on a site. Let us stress the following: A sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3 \to 0$$

is exact if and only if the sequence $0 \to \mathcal{F}_1(Y) \to \mathcal{F}_2(Y) \to \mathcal{F}_3(Y)$ is exact for any $Y \in \mathcal{C}$ and for any $s \in \mathcal{F}_3(Y)$ there exists a covering $\{Y_i \xrightarrow{f_i} Y\}_{i\in I}$ in τ, such that for all $i \in I$ the element $s_{|Y_i} = f_i^*(s) \in \mathcal{F}_3(Y_i)$ lies in the image of $\beta\colon \mathcal{F}_2(Y_i) \to \mathcal{F}_3(Y_i)$.

We want define étale cohomology as the right derived functor for global sections, and thus we have to ensure good homological properties for the category of abelian sheaves on the étale site of a scheme. For this, we add the smallness finiteness condition and arive at the very useful notion of a topos.

Definition 5.14. A topos $\mathfrak{X}$ is the category of sheaves $\mathrm{Sh}(\mathcal{C})$ on a site $\mathcal{C}$ satisfying the (Set-theoretic smallness) such that there exists a *set* $\mathcal{B} \subseteq \mathrm{ob}(\mathcal{C})$ of objects in $\mathcal{C}$ such that each covering $\{Y_i \to Y\}_{i\in I}$ can be refined by a covering $\{B_j \to Y\}_{j\in J}$ with $B_j \in \mathcal{B}$.

In essence, a topos is a (vast) sheaf-theoretic generalization of a topological space. The main examples that we will be interested in are the category $\widetilde{T} := \mathrm{Sh}(T)$ of sheaves on a topological space T and the étale topos $\widetilde{X_{\text{ét}}} := \mathrm{Sh}(X_{\text{ét}})$ of étale sheaves on a scheme X.

To streamline terminology, let us call the category $\mathrm{Sh}(\mathcal{C})$ of sheaves on a site $\mathcal{C}$ a topos as well. To highlight the set-theoretic conditions we call a topos medium if it is given by sheaves on some site satisfying the (set-theoretic smallness) for coverings, and small if the site additionally satisfies the conditions in Definition 5.14. In the remaining case, let us call a topos big. Let us stress that sheafification exists only for medium topoi. Usually, when speaking of a topos we mean a small one and write "topos" instead of "small topos".

Remark 5.15. Most often we will ignore the set theoretic issues from now on, and leave it to the context which implicit set-theoretic smallness conditions we assume!

Let us give a general construction for sheaves an a site.

[28]Such a τ is called a Grothendieck topology.

Example 5.16. Let $\mathcal{C}$ be a site and $Y \in \mathcal{C}$. Then the representable presheaf

$$h_Y := \mathrm{Hom}_{\mathcal{C}}(-, Y) \in \mathrm{PSh}(\mathcal{C})$$

need not be a sheaf in general. By the Yoneda lemma we have a naturally bijection

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_Y, \mathcal{F}) \cong \mathcal{F}(Y)$$

for any $\mathcal{F} \in \mathrm{PSh}(\mathcal{C})$. If $\mathcal{F}$ is a sheaf, we can conclude that

$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(h_Y^\sharp, \mathcal{F}),$$

where the "representable sheaf" $h_Y^\sharp$ is the sheafification of h_Y. If Λ is any ring, then for any sheaf of Λ-modules $\mathcal{F} \in \mathrm{Sh}_\Lambda(\mathcal{C})$ we can deduce the existence of a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(\Lambda[h_Y^\sharp], \mathcal{F}),$$

where $\Lambda[\mathcal{G}] := (Z \mapsto \Lambda[\mathcal{G}(Z)])^\sharp$ denotes the "free sheaf of Λ-modules" of the sheaf of sets $\mathcal{G} \in \mathrm{Sh}_\Lambda(\mathcal{C})$.

If each representable presheaf is a sheaf, then a site is called subcanonical. The main reason for putting the set-theoretic smallness conditions in the definition of a small topos is the following consequence.

Theorem 5.17. *Let $\mathfrak{X}$ be a (small) topos. Then for any ring Λ the category $\mathrm{Sh}_\Lambda(\mathfrak{X})$ of Λ-module objects is Grothendieck abelian. In particular, it has enough injectives and each complex admits a quasi-isomorphism to a K-injective complex ([Stacks, Tag 01D4]).*

The ring Λ can be replaced by any ring object $\mathcal{O}$ on $\mathfrak{X}$ by replacing $\Lambda[h_Y^\sharp]$ in the proof below by $(Z \mapsto \mathcal{O}(Z)[h_Y^\sharp(Z)])^\sharp$ (the "free $\mathcal{O}$-module on $h_Y^\sharp$").

Proof. Except for the existence of a generator all statements follow from the case of presheaves by sheafification. Let us write $\mathfrak{X} = \mathrm{Sh}(\mathcal{C})$ for some site (satisfying the required smallness assumptions). Let $\mathcal{B}$ be as in Definition 5.14. Then the sheaf

$$\bigoplus_{B \in \mathcal{B}} \Lambda[h_B^\sharp]$$

is a generator of $\mathrm{Sh}_\Lambda(\mathfrak{X})$. Indeed, if $\mathcal{F} \in \mathrm{Sh}_\Lambda(\mathfrak{C})$, then by Example 5.16 we obtain a natural morphism

$$\alpha\colon \bigoplus_{B \in \mathcal{B},\ s \in \mathcal{F}(B) \cong \mathrm{Hom}(\Lambda[h_B^\sharp], \mathcal{F})} \Lambda[h_B^\sharp] \xrightarrow{(s)} \mathcal{F},$$

which is a surjection because for every section $s \in \mathcal{F}(Y), Y \in \mathcal{C}$ we can refine the cover $\{Y \to Y\}$ by a cover $\{B_i \to Y\}_{i \in I}, B_i \in \mathcal{B}$, and then the restrictions $s_{|B_i}, i \in I$, lies in the (presheaf) image of α. ☐

The surjection α motivates the following more precise version.

Lemma 5.18. *Let $\mathcal{C}$ be a site and $\mathcal{F} \in \mathrm{Sh}(\mathcal{C})$. Then the map*

$$\mathcal{G} := \varinjlim_{Y \in \mathcal{C},\ s \in \mathcal{F}(Y) \cong \mathrm{Hom}(h_Y^\sharp, \mathcal{F})} h_Y^\sharp \to \mathcal{F}$$

is a isomorphism, which is natural in $\mathcal{F}$.

Proof. The statement for $\mathrm{Sh}(\mathcal{C})$ follows from the one for $\mathrm{PSh}(\mathcal{C})$ by sheafification. Hence, assume that $\mathcal{C}$ is just a category, and $\mathcal{F} \in \mathrm{PSh}(\mathcal{C})$. By the Yoneda lemma it suffices show that for any $\mathcal{H} \in \mathrm{PSh}(\mathcal{C})$ the map

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\mathcal{F}, \mathcal{H}) \to \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\mathcal{G}, \mathcal{H}) \cong \varprojlim_{Y \in \mathcal{C},\ s \in \mathcal{F}(Y)} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_Y, \mathcal{H}) \cong \varprojlim_{Y \in \mathcal{C},\ s \in \mathcal{F}(Y)} \mathcal{H}(Y)$$

is a bijection. But starring at this last inverse limits reveals that it specifies exactly a natural transformation $\eta\colon \mathcal{F} \to \mathcal{H}$. ☐

5.19. **Functoriality for sheaves.** In this section we want to discuss the notion of a "morphism between topoi". The basic example for topoi are the category of sheaves on a topological space, or the category of étale sheaves on a scheme. Assume that $f\colon Y \to X$ is a morphism of schemes. Then we obtain a pullback functor

$$f^{-1}_{\mathrm{sites}}\colon X_{\text{ét}} \to Y_{\text{ét}},\ (Z \to X) \mapsto (Z \times_X Y \to Y)$$

on étale sites, which preserves finite limits and maps an étale covering $\{Z_i \to Z\}_{i\in I}$ to the étale covering $\{Z_i \times_X Y \to Y\}_{i\in I}$. This implies that the functor

$$f^P_*\colon \mathrm{PSh}(Y_{\text{ét}}) \to \mathrm{PSh}(X_{\text{ét}}),\ F \mapsto \mathcal{F} \circ f^{-1}_{\mathrm{sites}}$$

induces a functor

$$f_*\colon \widetilde{Y_{\text{ét}}} \to \widetilde{X_{\text{ét}}}.$$

From the case of topological spaces we'd like to have that this functor has a left adjoint f^{-1}, *which is exact, i.e., commutes with finite limits.* Note that this commutation with finite limits is important, e.g., it implies f^{-1} preserves abelian sheaves/sheaves of rings/... and defines an exact functor for such. Motivated by this we make the following definition.

Definition 5.20. A morphism $f\colon \mathfrak{Y} \to \mathfrak{X}$ of topoi is a functor $f_*\colon \mathfrak{Y} \to \mathfrak{X}$, which admits an exact left adjoint $f^{-1}\colon \mathfrak{X} \to \mathfrak{Y}$. If $g\colon \mathfrak{Z} \to \mathfrak{Y},\ f\colon \mathfrak{Y} \to \mathfrak{X}$ are morphisms of topoi, then their composition $f \circ g$ is given by the functor $f_* \circ g_*$, whose left adjoint $g^{-1} \circ f^{-1}$ is indeed exact.

As equality of functors is badly behaved for categories it is better to also keep track of the natural transformations $\eta\colon f_* \to f'_*$ for two morphisms $f, f'\colon \mathfrak{Y} \to \mathfrak{X}$ of topoi. Thus topoi form naturally a 2-category. We will not make this point more precise as it is a bit orthogonal to the aims of this lecture.

The next lemma lets us construct many examples for morphisms of topoi.

Lemma 5.21. *Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor and define the functor*

$$u^{\mathrm{PSh}}_*\colon \mathrm{PSh}(\mathcal{D}) \to \mathrm{PSh}(\mathcal{C}),\ F \mapsto F \circ u.$$

1. *The functor u^{PSh}_* admits a left adjoint $u^{-1}_{\mathrm{PSh}}\colon \mathrm{PSh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{D})$.*
2. *If $G \in \mathrm{PSh}(\mathcal{C})$, then $u^{-1}_{\mathrm{PSh}}(G)$ is the "pointwise left Kan extension" of G along u, i.e.,*
$$(Z \in \mathcal{D}) \mapsto \varinjlim_{\{Y\in\mathcal{C},\ Z\to u(Y)\}^{\mathrm{op}}} G(Y).$$
3. *If $Y \in \mathcal{C}$ with contravariant Hom-functor $h_Y \in \mathrm{PSh}(\mathcal{C})$, then*
$$u^{-1}_{\mathrm{PSh}}(h_Y) = h_{u(Y)}.$$
4. *Assume that $\mathcal{C}$ has finite limits, and u preserves these. Then u^{-1}_{PSh} is exact. In particular, u^{PSh}_* defines a morphism $u^{\mathrm{PSh}}\colon \mathrm{PSh}(\mathcal{D}) \to \mathrm{PSh}(\mathcal{C})$ of (usually big) topoi.*

We are making implicit set-theoretic assumptions on $\mathcal{C}, \mathcal{D}$ here, namely that the colimits in (2) exists for any $Z \in \mathcal{D}$. Before starting the proof let us consider the example

$$u = f^{-1}\colon \mathrm{Ouv}(S) \to \mathrm{Ouv}(T)$$

for a continuous map $f\colon T \to S$ of topological spaces, and $\mathrm{Ouv}(T), \mathrm{Ouv}(S)$ the categories of open sets. Then

$$u^{-1}_{\mathrm{PSh}}(\mathcal{G})(U) = \varinjlim_{V\subseteq S \text{ open},\ U\subseteq f^{-1}(V)} \mathcal{G}(V)$$

is exactly the presheaf pullback that we have encountered in Algebraic Geometry I.

Proof. We can define u^{-1}_{PSh} by the formula in (2). Then it is straightforward to check that u^{-1}_{PSh} is left adjoint to u^{PSh}_* by identifying elements in $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{D})}(u^{-1}_{\mathrm{PSh}}(\mathcal{G}), \mathcal{F})$ and $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\mathcal{G}, u^{\mathrm{PSh}}_*(\mathcal{F}))$ with systems of maps $\varphi_{Y,Z,f}\colon \mathcal{G}(Y) \to \mathcal{F}(Z)$ for any $Y \in \mathcal{C}, Z \in \mathcal{D}$ and morphism $f\colon Z \to u(Y)$, which are compatible for varying the data Y, Z, f.

Let us show (3). Take $Y \in \mathcal{C}$ and $\mathcal{G} \in \mathrm{PSh}(\mathcal{D})$. Then

$$\begin{aligned} & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{D})}(u^{-1}_{\mathrm{PSh}}(h_Y), \mathcal{G}) \\ \overset{(2)}{=} & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_Y, u^{\mathrm{PSh}}_*(\mathcal{G})) \\ \overset{Yoneda}{=} & u_*(\mathcal{G})(Y) \\ = & \mathcal{G}(u(Y)) \\ \overset{Yoneda}{=} & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{D})}(h_{u(Y)}, \mathcal{G}). \end{aligned}$$

By a third application of the Yoneda lemma we can conclude that $u_{\mathrm{PSh}}^{-1}(h_Y)$ and $h_{u(Y)}$ are naturally isomorphic. Finally let us prove (4). From the pointwise construction of u_{PSh}^{-1} it suffices to check that for each $Z \in \mathcal{D}$ the category

$$I_Z := \{Y \in \mathcal{C},\ Z \to u(Y)\}^{\mathrm{op}}$$

is filtered. But as $\mathcal{C}$ has finite limits and u commutes with these, it follows easily that I_Z has finite colimits. But any category with finite colimits is filtered. □

Definition 5.22. Let $\mathcal{C}, \mathcal{D}$ be sites and let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor. Assume that

(1) the category $\mathcal{C}$ has finite limits and u commutes with these.
(2) if $\{Y_i \to Y\}_{i\in I}$ is a covering in $\mathcal{C}$, then $\{u(Y_i) \to u(Y)\}_{i\in I}$ is a covering in $\mathcal{D}$.

Then we set

$$f_*\colon \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C}),\ \mathcal{F} \mapsto u_*^{\mathrm{PSh}}(\mathcal{F}) = \mathcal{F} \circ u$$

and

$$f^{-1}\colon \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{D}),\ \mathcal{G} \mapsto (u_{\mathrm{PSh}}^{-1}(\mathcal{G}))^\sharp,$$

and call the associated morphism $f\colon \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C})$ of topoi the morphism associated with u.

5.23. **Examples of topoi and morphisms of topoi.** It is time to discuss some examples of topoi and morphisms of topoi.

(1) If T is a topological space, then $\mathrm{Sh}(T)$ is a topos. If $f\colon T \to S$ is continuous map, then we get a morphism of topoi
$$f\colon \mathrm{Sh}(T) \to \mathrm{Sh}(S)$$
given by the usual functor f_*. In particular, the category of sets is a topos as it identifies with $\mathrm{Sh}(\{*\})$.
(2) If X is a scheme. Then $\widetilde{X_{\text{ét}}} = \mathrm{Sh}(X_{\text{ét}})$ is a (small) topos. If $f\colon Y \to X$ is a morphism of schemes, then we get a morphism of topoi
$$f\colon \widetilde{Y_{\text{ét}}} \to \widetilde{X_{\text{ét}}}$$
with $f_*(\mathcal{F})(Z) := \mathcal{F}(Y \times_X Z)$ for $\mathcal{F} \in \mathrm{Sh}(Y_{\text{ét}})$ and $Z \in X_{\text{ét}}$. Namely, we can use Definition 5.22 for the functor $f_{\mathrm{sites}}^{-1}\colon X_{\text{ét}} \to Y_{\text{ét}},\ Z \mapsto Y \times_X Z$.
(3) Let $\mathcal{C}$ be a category. Then $\mathrm{PSh}(\mathcal{C})$ is a topos, cf. Remark 5.15. If $\mathcal{C}$ has finite limits and $u\colon \mathcal{C} \to \mathcal{D}$ is a functor preserving finite limits, then we get a morphism
$$f\colon \mathrm{PSh}(\mathcal{D}) \to \mathrm{PSh}(\mathcal{C})$$
of topoi with $f_* = u_*^{\mathrm{PSh}}$.
(4) Let $\mathcal{C}$ be a site and $\mathrm{Sh}(\mathcal{C})$ its topos of sheaves. Then the inclusion $i_*\colon \mathrm{Sh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ defines a morphism $i\colon \mathrm{Sh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ of topoi with $i^{-1} = (-)^\sharp$ given by sheafification.
(5) Let G be a group and let BG be the category with one object $*$ such that $\mathrm{Hom}_{BG}(*,*) = G$. Then
$$\mathrm{PSh}(BG) \cong (G - \mathrm{Sets}),\ \mathcal{F} \mapsto \mathcal{F}(*),$$
with G-action on $\mathcal{F}(*)$ induced by functoriality.
(6) Let G be a topological group (usually profinite) and let $\mathcal{C}$ be the category of continuous G-sets, i.e., sets S with an action of G such that the map $G \times S \to S$ is continuous when S is given the discrete topology. We equip $\mathcal{C}$ with the Grothendieck topology given by jointly surjective families of maps of G-sets. Then $\mathcal{C}$ is a site and $\mathrm{Sh}(\mathcal{C}) \cong \mathcal{C}$. Indeed, to $\mathcal{F} \in \mathrm{Sh}(\mathcal{C})$ we can associate the G-set $\varinjlim\limits_{H \subseteq G \text{ open}} \mathcal{F}(G/H)$, and to a continuous G-set S one can associate the sheaf $h_S = \mathrm{Hom}_{G-\mathrm{Sets}}(-, S)$. We write $G - \mathrm{Sets}^{\mathrm{cont}}$ for this topos of continuous G-sets. We will call this the classifying topos for G and denote it by BG (or $B_{\mathrm{cont}}G$ to highlight the topology on G).
(7) Assume that $X = \mathrm{Spec}(k)$ for a field k and let $\overline{k}$ be a separable closure of k. Then $X_{\text{ét}} \cong \mathrm{Gal}(\overline{k}/k) - \mathrm{Sets}^{\mathrm{cont}}$, where $\mathrm{Gal}(\overline{k}/k)$ is given the Krull topology.
(8) Let $\mathfrak{X}$ be any topos. Then there exists a unique (up to unique isomorphism) morphism $f\colon \mathfrak{X} \to \mathrm{Sets}$, i.e., Sets is the "terminal topos". Indeed, let $* \in \mathfrak{X}$ be a terminal object (=constant presheaf with value a one point set $\{*\}$). Then
$$f_* := \Gamma(\mathfrak{X}, -) := \mathrm{Hom}_{\mathfrak{X}}(*, -)\colon \mathfrak{X} \to \mathrm{Sets}$$
has an exact left adjoint given by the functor sending a set $S \in \mathrm{Sets}$ to the "constant sheaf with value S", denoted $\underline{S}$, which is given by the sheafification of the constant presheaf
$$Y \in \mathcal{C} \mapsto S$$

if $\mathfrak{X} = \mathrm{Sh}(\mathcal{C})$. The adjunction between $\underline{(-)}$ and $\Gamma(\mathfrak{X}, -)$ is easily checked using the universal property of sheafification. As sheafification is exact we also see that $\underline{(-)}$ is exact. Assume now that $g\colon \mathfrak{X} \to (\mathrm{Sets})$ is some morphism of topoi. Then $g^{-1}\colon (\mathrm{Sets}) \to \mathfrak{X}$ is a functor commuting with all colimits and finite limits. As it commutes with colimits (in particular with coproducts) it is uniquely determined by $g^{-1}(\{*\})$. But as $\{*\}$ is a terminal object in Sets, $g^{-1}(\{*\}) \in \mathfrak{X}$ must be a terminal object. But this already implies that $g^{-1} \cong \underline{(-)}$ as claimed.

(9) Let $\mathfrak{X}$ be a topos and $f\colon \mathcal{X} \to (\mathrm{Sets})$ the natural morphism with $f^{-1}(-) = \underline{(-)}$ the pullback. If Λ is any ring we conclude that $\underline{\Lambda}$ is a ring object in $\mathfrak{X}$ and the category of $\underline{\Lambda}$-module objects in $\mathfrak{X}$ identifies with the category of sheaves of Λ-modules on $\mathcal{C}$.

(10) If $\mathfrak{X} = \mathrm{Sh}(T)$ in the previous point for some topological space, then the representable sheaf $h_T \in \mathrm{Sh}(T)$ defines a terminal object and

$$\Gamma(\mathfrak{X}, -) = \Gamma(T, -).$$

Similarly, if $\mathfrak{X} = \widetilde{X_{\text{ét}}}$, then

$$\Gamma(\mathfrak{X}, -) \cong \Gamma(X, -)\colon \mathrm{Sh}(X_{\text{ét}}) \to (\mathrm{Sets}),\ \mathcal{F} \mapsto \mathcal{F}(X)$$

as h_X defines a terminal object in $\widetilde{X_{\text{ét}}}$.

(11) Let $\mathfrak{X}$ be a (small) topos and Λ be any ring. The right derived functors $H^i(\mathfrak{X}, -)$ of the functor $\Gamma(\mathfrak{X}, -)\colon \mathrm{Sh}_\Lambda(\mathfrak{X}) \to \mathrm{Mod}_\Lambda$ on the category of sheaves of Λ-modules, which exist by Theorem 5.17, are the cohomology functors of the topos $\mathfrak{X}$. In particular, if $\mathfrak{X} = \widetilde{X_{\text{ét}}}$ is the category of étale sheaves on a scheme, then

$$H^i_{\text{ét}}(X, -) := H^i(\widetilde{X_{\text{ét}}}, -),\ i \geq 0$$

define the étale cohomology groups of X. Understanding these better is the main aim of course.

(12) Assume $\mathfrak{X} = G - \mathrm{Sets}$ for some (abstract) group G. Then $\Gamma(\mathfrak{X}, -) = (-)^G$ is the functor of G-invariants on $G - \mathrm{Sets}$. Indeed, $(-)^G$ has the "trivial G-set" functor $S \mapsto S$ (with trivial G-action) as an exact left adjoint, and hence must be isomorphic to $\Gamma(\mathfrak{X}, -)$ by (8). Hence, the cohomoloy of $\mathfrak{X}$ is given by group cohomology of G.

(13) Assume that G is a profinite group. Then similar to (12) we see that $\Gamma(G - \mathrm{Sets}^{\mathrm{cont}}, -)$ calculates the continuous group cohomology of G for discrete G-modules. Combining with (7) and (8) we see that étale cohomoloy of fields is exactly given by Galois cohomology.

(14) Let S be a scheme. Then $\mathcal{C} := (\mathrm{Sch}/S)$ can be given the following, increasingly fine, Grothendieck topologies, where a familiy $\{f_i\colon Y_i \to Y\}_{i\in I}$ is a covering if
- (the Zariski topology) the f_i are jointly surjective open immersions,
- (the étale topology) the f_i are jointly surjective and étale,
- (the fppf-topology) the f_i are jointly surjective flat and locally of finite presentation,
- (the fpqc-topology) there exists a refinement[29] $\{g_j\colon Z_j \to Y\}_{j\in J}$ of $\{f_i\colon Y_i \to Y\}_{i\in I}$ such that $\coprod_{j\in J} Z_j \to Y$ is faithfully flat *and quasi-compact*.

To relate the first three topologies to the last one, we actually need a theorem.

Theorem 5.24. *Let $f\colon W \to V$ be a flat morphism of schemes, which is locally of finite presentation. Then f is (universally) open.*

Proof. We'll probably discuss a proof of this theorem later. For now we give a reference to [Stacks, Tag 01U1]. □

In the first three examples, the morphisms f_i are therefore (universally) open, and this implies that they are also fpqc-covers. Indeed, one may reduce to the case that Y is quasi-compact, and then finitely many of open sets $f_i(Y_i)$ cover Y. We get morphisms of topoi[30]

$$\mathrm{Sh}(\mathcal{C}_{\mathrm{fpqc}}) \to \mathrm{Sh}(\mathcal{C}_{\mathrm{fppf}}) \to \mathrm{Sh}(\mathcal{C}_{\mathrm{etale}}) \to \mathrm{Sh}(\mathcal{C}_{\mathrm{Zar}}).$$

Let us note that the fact that faithfully flat, quasi-compact maps are topological quotient maps implies that for any topological space (e.g., discrete) T the presheaf

$$S \mapsto \mathrm{Hom}_{\mathrm{cont}}(S, T)$$

[29]This condition has to made in order to avoid families like $\{\mathrm{Spec}(\mathbb{Z}_{(p)}) \to \mathrm{Spec}(\mathbb{Z})\}_{p \text{ prime}}$.

[30]Actually, for the fpqc-topology the set-theoretic issues are relevant. While it is perfectly fine to speak about fpqc-sheaves, we should avoid speaking about sheafification for the fpqc-topology, or cohomology. One cannot avoid these issues by choosing a cut-off cardinal κ as the resulting sheafification/cohomoloy does in general depend on κ.

is an fpqc-sheaf, and hence a fortiori an étale sheaf. We will prove in Corollary 7.14 that for each scheme X over S the representable functor h_X is an fpqc-sheaf.

(15) With the notation from (14) there exists (using Lemma 5.21) natural morphisms of topoi

$$\varepsilon\colon \mathrm{Sh}(\mathcal{C}_{\mathrm{etale}}) \to \widetilde{S_{\text{ét}}},\ \eta\colon \mathrm{Sh}(\mathcal{C}_{\mathrm{Zar}}) \to \widetilde{S_{\mathrm{Zar}}}$$

induced by the inclusion $u\colon S_{\text{ét}} \to (\mathrm{Sch}/S)$ resp. $u\colon S_{\mathrm{Zar}} \to (\mathrm{Sch}/S)$. Note that ε_*, η_* are exact as any covering in Sch/S of some $Y \in S_{\text{ét}}$ or S_{Zar} is actually induced by a covering in $S_{\text{ét}}$ resp. S_{Zar} (the functor "u is cocontinuous").

(16) Let $\mathcal{C}$ be the category of profinite sets[31] and equip it with the Grothendieck topology with coverings $\{S_i \to S\}_{i\in I}$ such that there exists a *finite* subset $J \subseteq I$ such that $\coprod_{i\in J} S_i \to S$ is surjective. The associated topos $\mathrm{Cond} := \mathrm{Sh}(\mathcal{C})$ is the topos of condensed sets studied in [25]. Let T be any topological space, e.g., $\mathbb{R}$. Then

$$S \mapsto \underline{T}(S) := \mathrm{Hom}_{\mathrm{cont}}(S, T)$$

defines a condensed set, and the resulting functor Top $\to$ Cond is fully faithful on compactly generated topological spaces, e.g., metric or locally compact.

After having discussed examples of topoi and morphisms between them, we turn to some abstract constructions with topoi.

5.25. **Abstract constructions with topoi.** Let us now discuss some abstract constructions with topoi. If $\mathcal{C}$ is a category and $Y \in \mathcal{C}$, then we recall that $\mathcal{C}/Y$ denotes the category of arrows $f_Z\colon Z \to Y$ with $Z \in \mathcal{C}$, and morphisms given by morphisms $g\colon Z_1 \to Z_2$ such that $f_{Z_2} \circ g = f_{Z_1}$.

Assume now that $\mathcal{C}$ is a site.

Definition 5.26. We define the site $\mathcal{C}/Y$ by requiring that a collection $\{f_i\colon Z_i \to Z\}_{i\in I}$ of morphisms in $\mathcal{C}/Y$, i.e., we are given implicit structure maps $Z_i \to Y, Z \to Y$ and all f_i respect these, is a covering if and only if it is a covering in $\mathcal{C}$, i.e., after forgetting the structure morphism to Y.

Example 5.27. If X is a scheme and $Y \to X$ étale, then $X_{\text{ét}}/Y \cong Y_{\text{ét}}$ by definition. Similarly, if T is a topological space and $U \subseteq T$ open, then $\mathrm{Ouv}(T)/U \cong \mathrm{Ouv}(U)$.

Motivated by this example we would like $\mathcal{C}/Y$ as some sort of "open subset" of $\mathcal{C}$. From a sheaf theoretic perspective this motivates the desire of functors

$$j^*\colon \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C}/Y),\ j_!, j_*\colon \mathrm{Sh}(\mathcal{C}/Y) \to \mathrm{Sh}(\mathcal{C})$$

with the "extension by the empty set functor" $j_!$ left adjoint to j^* and j^* left adjoint to j_*.

For simplicity we assume from now on that $\mathcal{C}$ has all finite limits.[32]

Let $v\colon \mathcal{C}/Y \to \mathcal{C}$, $(f_Z\colon Z \to Y) \mapsto Z$ be the natural functor. By the universal property of products, it has the functor

$$u\colon \mathcal{C} \to \mathcal{C}/Y,\ Z \mapsto (Z \times Y \to Y)$$

as a right adjoint. This implies formally that (in the notation of Lemma 5.21)

$$u^{-1}_{\mathrm{PSh}} \cong v^{\mathrm{PSh}}_*$$

because the functor v^{PSh}_* is left adjoint to u^{PSh}_*.[33]

From the definition of coverings in $\mathcal{C}/Y$ we see that u^{PSh}_* and v^{PSh}_* preserve sheaves. Hence, we can make the following definitions.

Definition 5.28. In the above situation, we set

- $j^* := v^{\mathrm{PSh}}_*\colon \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C}/Y)$, $\mathcal{F} \mapsto ((Z \to Y) \mapsto \mathcal{F}(Z))$,
- $j_* := u^{\mathrm{PSh}}_*\colon \mathrm{Sh}(\mathcal{C}/Y) \to \mathrm{Sh}(\mathcal{C})$, $\mathcal{G} \mapsto (Z \mapsto \mathcal{G}(Z \times Y)$,
- $j_! := (-)^\sharp \circ v^{-1}_{\mathrm{PSh}}\colon \mathrm{Sh}(\mathcal{C}/Y) \to \mathrm{Sh}(\mathcal{C})$,
- $j = (j^*, j_*)\colon \mathrm{Sh}(\mathcal{C}/Y) \to \mathrm{Sh}(\mathcal{C})$, which is a morphism of topoi (but $(j_!, j^*)$ in general not!).

If $(Z \to Y) \in \mathcal{C}/Y$ and $h^\sharp_{(Z\to Y)} \in \mathrm{Sh}(\mathcal{C}/Y)$ is the associated representable sheaf, then

$$j_!(h^\sharp_{(Z\to Y)}) = h^\sharp_Z$$

[31] For set-theoretic issues one should bound their size by some cut-off cardinal.

[32] This is not necessary, cf. [Stacks, Tag 00XZ].

[33] This follows formally from the statement that the 2-functor $\mathrm{Fun}(-, \mathrm{Sets})$ preserves adjunctions as these can be defined via the triangle identities.

by Lemma 5.21. As $j_!$ commutes with all colimits, this characterizes $j_!$ uniquely by Lemma 5.18. In particular, we see that $j_!$ maps the terminal object $(Y \to Y) \in \mathcal{C}/Y$ to the representable sheaf $h_Y^\sharp$. This implies that $j_!$ induces a functor

$$\widetilde{j}_! \colon \mathrm{Sh}(\mathcal{C}/Y) \to \mathrm{Sh}(\mathcal{C})/h_Y^\sharp.$$

Lemma 5.29. *The functor*

$$\widetilde{j}_! \colon \mathrm{Sh}(\mathcal{C}/Y) \to \mathrm{Sh}(\mathcal{C})/h_Y^\sharp$$

is an equivalence.

Proof. A quasi-inverse is given by the functor

$$(\mathcal{F} \to h_Y^\sharp) \mapsto ((Z \to Y) \mapsto \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})/h_Y^\sharp}((h_Z^\sharp \to h_Y^\sharp), (\mathcal{F} \to h_Y^\sharp)).$$

Details can be found in [Stacks, Tag 00Y1]. □

As a consequence we get the following.

Lemma 5.30. *The functor* $j_! \colon \mathrm{Sh}(\mathcal{C}/Y) \to \mathrm{Sh}(\mathcal{C})$ *commutes with equalizers and fiber products.*

Proof. By Lemma 5.29 this reduces to the following easy observation: For any category $\mathcal{D}$ and $Z \in \mathcal{D}$ the functor $\mathcal{D}/Z \to \mathcal{D}$ commutes with equalizers and fiber products. □

Remark 5.31. In general, the natural maps $\mathcal{G} \to j^*j_!(\mathcal{G})$, $j^*j_*\mathcal{G} \to \mathcal{G}$ are *not* isomorphisms, and thus the analogy of $\mathrm{Sh}(\mathcal{C}/Y)$ being sheaves of an "open subset" of $\mathrm{Sh}(\mathcal{C})$ breakes down. Indeed, for $(Z \to Y) \in \mathcal{C}/Y$ we have $j^*j_*(\mathcal{G})(Z \to Y) = j_*(\mathcal{G})(Z) = \mathcal{G}(Z \times Y \to Y) \neq \mathcal{G}(Z \to Y)$ in general (and similarly for $j^*j_!$).

Exercise 5.32.
(1) Let $* \in \mathcal{C}$ be a terminal object and assume that the canonical morphism $Y \to *$ is a monomorphism. Then the natural maps
$$\mathcal{G} \to j^*j_!(\mathcal{G}),\ j^*j_*\mathcal{G} \to \mathcal{G}$$
are isomorphisms.
(2) Let T be a topological space and set $* := h_T \in \widetilde{T} = \mathrm{Sh}(T)$. Show that the set of isomorphism classes of monomorphisms $Y \to *$ with $Y \in \widetilde{T}$ is in natural bijection with the set of open subsets of T.

Next, we want to clarify (in the form of an exercise) a bit the relation between sites and topoi.

Exercise 5.33. Let $\mathfrak{X}$ be a topos. The canonical topology on $\mathfrak{X}$ is the Grothendieck topology with $\{Y_i \to Y\}_{i\in I}$ a covering if and only if $\coprod\limits_{i\in I} Y_i \to Y$ is an epimorphism.

(1) Show that this defines indeed a Grothendieck topology.
(2) Show that a functor $\mathcal{F} \colon \mathfrak{X}^{\mathrm{op}} \to (\mathrm{Sets})$ is a sheaf for the canonical topology if and only if commutes with limits.
(3) If $\mathfrak{X}$ is small, show that a functor $\mathcal{F} \colon \mathfrak{X}^{\mathrm{op}} \to (\mathrm{Sets})$ is a sheaf if and only if $\mathcal{F} \cong h_Y$ for some $Y \in \mathfrak{X}$. *Hint: Use the general statement [Stacks, Tag 0AHN] or some other form of the adjoint functor theorem.*
(4) If $\mathfrak{X} \cong \mathrm{Sh}(\mathcal{C})$ for some site $\mathcal{C}$ and $\{Y_i \to Y\}_{i\in I}$ is a collection of morphisms in $\mathcal{C}$, then $\{h_{Y_i}^\sharp \to h_Y^\sharp\}_{i\in I}$ is a covering in $\mathfrak{X}$ if and only if there exists a refinement $\{Z_j \to Y\}_{j\in J}$ if $\{Y_i \to Y\}_{i\in I}$, which is a covering in $\mathcal{C}$.

In particular, $\mathfrak{X} \cong \mathrm{Sh}(\mathfrak{X})$ via the Yoneda embedding. The next exercise can be used to produce many different sites with equivalent topoi.

Exercise 5.34. Let $\mathcal{C}$ be a site with finite limits and let $\mathcal{B} \subseteq \mathcal{C}$ be a full subcategory, stable under finite limits. Assume that for each $Y \in \mathcal{C}$ and each covering $\{Y_i \in Y\}_{i\in I}$, there exists a cover $\{B_j \to Y\}_{j\in J}$ refining $\{Y_i \to Y\}_{i\in I}$ such that $B_j \in \mathcal{B}$ for all $j \in J$.

(1) Let the covers in $\mathcal{B}$ be those collections of morphisms, which define a covering in $\mathcal{C}$. Show that this defines a Grothendieck topology on $\mathcal{B}$.
(2) Let $u \colon \mathcal{B} \to \mathcal{C}$ be the inclusion, with associated morphism of topoi $(u^{-1}, u_*) \colon \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{B})$. Show that for any $\mathcal{G} \in \mathrm{Sh}(\mathcal{B})$ the natural maps
$$\mathcal{G} \to u_*^{\mathrm{PSh}} \circ u_{\mathrm{PSh}}^{-1}(\mathcal{G}) \to u_* \circ u^{-1}(\mathcal{G})$$
are isomorphisms. In particular, u^{-1} is fully faithful. *Hint: Use Lemma 5.7.*
(3) Show that for any $\mathcal{F} \in \mathrm{Sh}(\mathcal{C})$ there exists $\mathcal{G} \in \mathrm{Sh}(\mathcal{B})$ and an epimorphism $u^{-1}\mathcal{G} \to \mathcal{F}$. *Hint: Ignore set theoretic difficulties and use a surjection as in Theorem 5.17.*

(4) Show that u^{-1} is essentially surjective, and thus an equivalence of topoi. *Hint: Try to represent* $\mathcal{F} \in \mathrm{Sh}(\mathcal{C})$ *as a coequalizer of objects in the essential image of* u^{-1}.

Exercise 5.34 can be applied in various situations.

Example 5.35. The following are instances of Exercise 5.34.

(1) Let T be a topological space, and $\mathcal{B}$ a basis for the topology of T. Then $\mathrm{Sh}(T)$ is equivalent to the category of sheaves on the basis $\mathcal{B}$, cf. [Stacks, Tag 009H].
(2) Let X be a scheme and $X_{\text{ét}}^{\text{aff}} \subseteq X_{\text{ét}}$ the subcategory of those $f\colon Y \to X$ with Y an affine scheme and $f(Y)$ contained in some affine of X. Then

$$\mathrm{Sh}(X_{\text{ét}}^{\text{aff}}) \cong \mathrm{Sh}(X_{\text{ét}}).$$

5.36. **Cohomology of topoi.** In this section all topoi are assumed to be small. We first introduce the topos theoretic analog of a "ringed topological space".

Definition 5.37. A ringed topos is a pair $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ of a topos and a ring object $\mathcal{O}_{\mathfrak{X}} \in \mathfrak{X}$. A morphism $f\colon (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ of ringed topoi is a pair of a morphism $f_0\colon \mathfrak{Y} \to \mathfrak{X}$ of topoi and a morphism $f^\sharp\colon \mathcal{O}_{\mathfrak{X}} \to f_{0,*}\mathcal{O}_{\mathfrak{Y}}$ of ring objects in $\mathfrak{X}$.

The composition is defined as for ringed spaces. Note that equivalently, we can view $f^\sharp$ as a morphism

$$f_0^{-1}\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{Y}}.$$

For us the most important example is $\mathcal{O}_{\mathfrak{X}} = \underline{\Lambda}$ for some ring Λ. Note that any morphism $f_0\colon \mathfrak{Y} \to \mathfrak{X}$ naturally upgrades to a morphism $f\colon (\mathfrak{Y}, \underline{\Lambda}) \to (\mathfrak{X}, \underline{\Lambda})$ by setting $f^\sharp$ as the adjoint to the natural isomorphism

$$f_0^{-1}\underline{\Lambda} \cong \underline{\Lambda}.$$

Often we suppress $\mathcal{O}_{\mathfrak{X}}$ from the notation $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. Given a ringed topos $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ we set

$$\mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}}}$$

as the (Grothendieck) abelian category of $\mathcal{O}_{\mathfrak{X}}$-module objects in $\mathfrak{X}$, and

$$D(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) := D(\mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}}})$$

as its derived category. Note that the following theorem is almost ridiculously general, e.g., by the examples in Section 5.23 it deals with sheaf cohomology on topological spaces, étale cohomology for schemes, (continuous) group cohomology, derived inverse limits, condensed cohomology,...

Theorem 5.38. *Let* $f\colon \mathfrak{Y} \to \mathfrak{X}$ *be a morphism of ringed topoi.*

(1) *Set* $f_*\colon \mathrm{Mod}_{\mathcal{O}_{\mathfrak{Y}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}}}$, $\mathcal{F} \mapsto f_*\mathcal{F}$ *with* $\mathcal{O}_{\mathfrak{X}}$ *induced via* $f^\sharp\colon \mathcal{O}_{\mathfrak{X}} \to f_*(\mathcal{O}_{\mathfrak{Y}})$. *Then* f_* *admits a right derived functor*

$$Rf_*\colon D(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) \to D(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}),$$

and if $g\colon \mathfrak{Z} \to \mathfrak{Y}$ *is another morphism of ringed topoi, then* $f_* \circ g_* \cong (f \circ g)_*$ *and* $Rf_* \circ Rg_* \cong R(f \circ g)_*$.
(2) *Set* $f^*\colon \mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathfrak{Y}}}$, $\mathcal{G} \mapsto f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{Y}}$. *Then* f^* *admits a left derived functor*

$$Lf^*\colon D(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to D(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}),$$

and if $g\colon \mathfrak{Z} \to \mathfrak{Y}$ *is another morphism of ringed topoi, then* $g^* \circ f^* \cong (f \circ g)^*$ *and* $Lg^* \circ Lf^* \cong L(f \circ g)^*$.
(3) f^* *is left adjoint to* f_* *and* Lf^* *is left adjoint to* Rf_*.

Proof. We have discussed this theorem for topological spaces in the last term. Using Theorem 5.17 can transfer all arguments to this case, cf. [Stacks, Tag 01FQ]. Let us only mention two points: In the situations we are interested in, e.g., $\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{Y}}$ associated to some fixed ring Λ, the functor f^* is exact and thus passes directly to the derived category (thus developing a theory of K-flat complexes can be avoided for our purposes). The natural isomorphism $Rf_* \circ Rg_* \cong R(f \circ g)_*$ follows from $Lg^* \circ Lf^* \cong L(f \circ g)^*$ by adjunction. □

Implicitly, we have used in Theorem 5.38 that f^{-1}, f_* commute with finite limits (for sheaves of sets) and thus map sheaves of abelian groups, etc., to sheaves of abelian groups, etc. If $\mathfrak{Y} = \mathfrak{X}/Y$ and $f = j\colon \mathfrak{X}/Y \to \mathfrak{X}$ as in Lemma 5.29, then $j_!$ does not preserve finite products. Nevertheless there exists an exact left adjoint for j^* as we will check now.

Lemma 5.39. *Define the functor* $j_!\colon \mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}/Y}} \to \mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}}}$ *by sending* $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}/Y}}$ *to the sheafification of*

$$Z \in \mathfrak{X} \mapsto \bigoplus_{f \in \mathrm{Hom}_{\mathfrak{X}}(Z,Y)} \mathcal{G}(Z \to Y).$$

Then the functor $j_!$ *is exact and left adjoint to* $j^*\colon \mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}/Y}}$. *Moreover,*

$$j_!(\mathcal{O}_{\mathfrak{X}/Y}[h_{(Z\to Y)}]) \cong \mathcal{O}_{\mathfrak{X}}[h_Z]$$

for any $(Z \to Y) \in \mathfrak{X}/Y$.

Proof. Exactness follows by exactness of direct sums and sheafification. Adjointness of $j_!$ and j^* follows because $\mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}}}}(j_!\mathcal{G}, \mathcal{F})$, $\mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}/Y}}}(\mathcal{G}, j^*\mathcal{F})$ identify both with compatible systems of maps of $\mathcal{O}_{\mathfrak{X}}(Z) = j^*\mathcal{O}_{\mathfrak{X}/Y}(Z \to Y)$-modules

$$\mathcal{G}(Z \xrightarrow{f} Y) \to \mathcal{F}(Z)$$

for $Z \in \mathfrak{X}$ and $f \in \mathrm{Hom}_{\mathfrak{X}}(Z, Y)$. Now let $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}}}$. Then

$$\begin{array}{cc} & \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}}}}(j_!(\mathcal{O}_{\mathfrak{X}/Y}[h_{(Z\to Y)}]), \mathcal{F}) \\ = & \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}/Y}}}(\mathcal{O}_{\mathfrak{X}/Y}[h_{(Z\to Y)}], j^*\mathcal{F}) \\ = & j^*\mathcal{F}(Z \to Y) \\ = & \mathcal{F}(Z) \\ = & \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}}}}(\mathcal{O}_{\mathfrak{X}}[h_Z]), \mathcal{F}) \end{array}$$

as desired. □

We can now deduce that push forward can be calculated as expected.

Corollary 5.40. *Let* $f\colon \mathfrak{Y} \to \mathfrak{X}$ *be a morphism of topoi and* $Y \in \mathfrak{X}$. *Let* $j\colon \mathfrak{X}/Y \to \mathfrak{X}$, $j'\colon \mathfrak{Y}/f^{-1}(Y) \to \mathfrak{Y}$ *be the natural morphisms of topoi, cf. Definition 5.28*

1. *The functor* $(Z \to Y) \in \mathfrak{X}/Y \mapsto (f^{-1}(Z) \to f^{-1}(Y)) \in \mathfrak{Y}/f^{-1}(Y)$ *is the left adjoint* f_Y^{-1} *for a morphism* $f_Y\colon \mathfrak{Y}/f^{-1}(Y) \to \mathfrak{X}/Y$ *of topoi, such that* $j \circ f_Y \cong f \circ j'$.
2. $f_{Y,*}(Z \to f^{-1}(Y)) \cong (f_*(Z) \times_{f_*f^{-1}(Y)} Y \to Y)$ *for* $(Z \to f^{-1}(Y)) \in \mathfrak{Y}/f^{-1}(Y)$.
3. $j^*f_* \cong f_{Y,*}j'^{,*}$ *and* $j^*Rf_* \cong Rf_{Y,*}j'^{,*}$
4. *If* $K \in D(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ *and* $i \in \mathbb{Z}$, *then* $R^if_*(K) \in \mathrm{Mod}_{\mathcal{O}_{\mathfrak{X}}}$ *is the sheafification of the presheaf*

$$Y \in \mathfrak{X} \mapsto H^i(\mathfrak{Y}/f^{-1}(Y), j'^{,*}K).$$

Proof. The first two statements are formal. Let $Z \in \mathfrak{Y}$. Then

$$j^*(f_*(Z)) = (f_*(Z) \times Y \to Y) \in \mathfrak{X}/Y$$

and

$$f_{Y,*}j'(Z) = f_{Y,*}(Z \times f^{-1}Y \to f^{-1}Y) = (f_*(Z) \times f_*f^{-1}Y \times_{f_*f^{-1}(Y)} Y \to Y),$$

which implies that both are naturaly isomorphic. By Lemma 5.39 we can conclude that $j'^{,*}$ preserves K-injectives, and hence $R(f_{Y,*} \circ j'^{,*}) \cong Rf_{Y,*} \circ j'^{,*}$, which implies the rest of (3). For (4) assume that $K^\bullet$ is K-injective and $K^\bullet \cong K$. Then we calculate

$$R^if_*(K) = \mathcal{H}^i(f_*(K^\bullet)),$$

and this is the sheafification of $Y \mapsto \mathcal{H}^i(K^\bullet(f^{-1}(Y)))$ because $f_*(K^\bullet)(Y) = K^\bullet(f^{-1}(Y))$ (by adjunction between f^{-1} and f_*). Now $K^\bullet(f^{-1}(Y)) = (j'^{,*}K^\bullet)(f^{-1}(Y)) = \Gamma(\mathfrak{Y}/f^{-1}(Y), j'^{,*}K^\bullet)$ and by K-injectivity of $j'^{,*}K^\bullet$ (implied by Lemma 5.39) this last complex calculates $R\Gamma(\mathfrak{Y}/f^{-1}(Y), j'^{,*}K)$ as desired. □

After having discussed the general theory of cohomology for topoi let us come to the question of how to compute any étale cohomology group of some scheme.

6. The étale cohomology of proper smooth curves, I

Let k be an algebraically closed field. In this section we want to present the strategy of how to calculate the étale cohomology groups

$$H^*_{\text{ét}}(X,\Lambda)$$

for smooth, proper, connected curve X over k, e.g., $X=\mathbb{P}^1_k$, at least for certain abelian groups Λ.

We will do this by presenting possible strategy of calculating $H^*(X,\mathbb{Z})$ for a compact Riemann surface X (=compact complex manifold of dimension 1) via arguments close to results internal to complex analytic spaces. If $(X,\mathcal{O}_X)$ is a complex manifold, then we recall the exponential sequence

$$0\to \underline{\mathbb{Z}(1)}\to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0$$

on X, where $\underline{\mathbb{Z}(1)}$ is the constant sheaf associated to the abelian group $\mathbb{Z}\cdot 2\pi\cdot i\subseteq \mathbb{C}$.[34] We get the long exact sequence

$$0\to \underbrace{\mathbb{Z}(1)(X)}_{\cong \mathbb{Z}^{\pi_0(X)}} \to \mathcal{O}_X(X)\to \mathcal{O}_X^*(X)\to H^1(X,\mathbb{Z}(1))\to H^1(X,\mathcal{O}_X)\to \underbrace{H^1(X,\mathcal{O}_X^*)}_{\cong \mathrm{Pic}(X)} \xrightarrow{c_1}$$

$$\to H^2(X,\mathbb{Z}(1))\to H^2(X,\mathcal{O}_X)\to \ldots.$$

Here, we used that $\mathcal{O}_X^*$-torsors on X identify with line bundles on X. The connecting morphism

$$c_1\colon \mathrm{Pic}(X)\cong H^1(X,\mathcal{O}_X^*)\to H^2(X,\mathbb{Z}(1))$$

defines the first Chern class of a line bundle.

Now assume that $X\to \mathrm{Spec}(\mathbb{C})$ is a smooth, proper, connected curve. By Theorem 4.35 and Theorem 3.51 we can conclude that $H^n(X^{\mathrm{an}},\mathbb{Z}(1))=0$ for $n\geq 3$ and $H^n(X^{\mathrm{an}},\mathcal{O}_{X^{\mathrm{an}}})=0$ for $n\geq 2$. In particular, $H^2_{\text{ét}}(X^{\mathrm{an}},\mathcal{O}_X^*)=0$. Moreover, $H^1(X^{\mathrm{an}},\mathcal{O}^*_{X^{\mathrm{an}}})\cong H^1(X,\mathcal{O}_X^*)\cong \mathrm{Pic}(X)$ is the group of *algebraic* line bundles on X (again by Section 3.50). Thus, the long exact sequence simplifies to the exact sequence

$$0\to \mathbb{Z}(1)\to \mathbb{C}\xrightarrow{\exp}\mathbb{C}^*\xrightarrow{0} H^1(X^{\mathrm{an}},Z(1))\to H^1(X,\mathcal{O}_X)\to \underbrace{H^1(X,\mathcal{O}_X^*)}_{\cong\mathrm{Pic}(X)}\xrightarrow{c_1} H^2(X^{\mathrm{an}},\mathbb{Z}(1))\to 0\to\ldots$$

(using as well that $\exp\colon \mathbb{C}\to\mathbb{C}^\times$ is surjective). In particular, the group $H^1(X^{\mathrm{an}},\mathbb{Z}(1))$ is torsion free (and finitely generated by Exercise 4.43). We also know that the group $H^2(X^{\mathrm{an}},\mathbb{Z}(1))$ is finitely generated. The Picard group $\mathrm{Pic}(X)$ sits in an exact sequence

$$0\to \mathrm{Pic}^0(X)\to \mathrm{Pic}(X)\xrightarrow{\deg}\mathbb{Z}\to 0$$

with $\mathrm{Pic}^0(X)$ being a divisible group (the "Jacobian" of X). The map $\mathrm{Pic}^0(X)\to H^2(X^{\mathrm{an}},\mathbb{Z}(1))$ must therefore vanish as $\mathrm{Pic}^0(X)$ is n-divisible and $H^2(X^{\mathrm{an}},\mathbb{Z}(1))$ finitely generated. As the kernel of c_1 is a quotient of $H^1(X,\mathcal{O})$ and hence divisible, we see again by finite generation of $H^2(X^{\mathrm{an}},\mathbb{Z}(1))$ that $\ker(c_1)=\mathrm{Pic}^0(X)$ as $\mathrm{Pic}^0(X)$ is the largest divisible subgroup in $\mathrm{Pic}(X)$. This implies as well that $c_1=\pm\deg$.

We now prove the full description of the cohomology $H^*(X^{\mathrm{an}},\mathbb{Z}(1))\cong H^*(X^{\mathrm{an}},\mathbb{Z})$.

Theorem 6.1. *Let Λ be any abelian group. Let $g:=\dim H^1(X,\mathcal{O})$ be the genus of X. Then*

$$H^i(X^{\mathrm{an}},\Lambda)\cong\begin{cases}\Lambda, & i=0,2\\ \Lambda^{2g}, & i=1\\ 0, & i\geq 3.\end{cases}$$

Proof. We leave as an exercise to check that $H^*(X^{\mathrm{an}},\Lambda)\cong H^*(X^{\mathrm{an}},\mathbb{Z}(1))\otimes_{\mathbb{Z}}\Lambda$ (*Hint: Lemma 4.20 and $H^*(X^{\mathrm{an}},\mathbb{Z}(1))$ is torsion free.*). This reduces to the case that $\Lambda=\mathbb{Z}\cong\mathbb{Z}(1)$. By the above considerations it suffices to check that the Euler characteristic

$$\chi(X^{\mathrm{an}},\mathbb{C}):=\sum_{i=0}^{\infty}(-1)^i\dim_{\mathbb{C}} H^i(X^{\mathrm{an}},\mathbb{C})=2-\dim_{\mathbb{C}} H^1(X^{\mathrm{an}},\mathbb{C})$$

is equal to $2-2g$. The sequence

$$0\to\underline{\mathbb{C}}\to\mathcal{O}_X\xrightarrow{d}\Omega^1_{X^{\mathrm{an}}/\mathbb{C}}\to 0$$

(with d the differential) is exact as this can be checked locally on X^{an}, where X^{an} is isomorphic to an open ball in $\mathbb{C}$ and on those open balls holomorphic functions can be integrated. Now the Euler characteristic is additive in short exact sequences and using GAGA and Serre duality for X we can deduce

$$\chi(X^{\mathrm{an}},\mathbb{C})+(g-1)=1-g$$

[34] Thus, $\underline{\mathbb{Z}(1)}\cong\underline{\mathbb{Z}}$, but such an isomorphism depends on the choice of some element $i\in\mathbb{C}$ with $i^2=-1$.

as desired. □

With a bit more work one can show that $\mathbb{Z}^{2g} \cong H^1(X^{\mathrm{an}}, \mathbb{Z}(1)) \subseteq H^1(X, \mathcal{O}_X) \cong \mathbb{C}^g$ is a lattice, cf. [1], and deduce the structure of a complex torus on $\mathrm{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)/H^1(X^{\mathrm{an}}, \mathbb{Z}(1))$.

Of course, we'd like to prove a similar theorem in étale cohomolgy. However, we used something inherently analytic, namely the exponential.

Let $n \in \mathbb{Z}$ be non-zero integer. Then for $\Lambda = \mathbb{Z}/n$ we can give a much more algebraic proof of Theorem 6.1 and this will (with many difficulties on the way!) transport to étale cohomology.

Let us present this alternative calculation. The starting point is the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & \mathcal{O}_X & \xrightarrow{\exp} & \mathcal{O}_X^* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \exp(1/n(-))} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \mathbb{Z}/n(1) & \longrightarrow & \mathcal{O}_X^\times & \xrightarrow{n} & \mathcal{O}_X^* & \longrightarrow & 0 \end{array}$$

with $\mathbb{Z}/n(1) \cong \underline{\mathbb{Z}/n}$ the sheaf $U \mapsto \{z \in \mathcal{O}_X^*(U) \mid z^n = 1\}$ of n-th roots of unity. Taking the long exact sequence for the lower line we get by the following exact sequence

$$0 \to \underbrace{\mathbb{Z}}_{\cong \mathbb{Z}/n}/n(1) \to \mathbb{C}^\times \xrightarrow{n} \mathbb{C}^\times \xrightarrow{0} H^1(X^{\mathrm{an}}, \mathbb{Z}/n(1)) \to \mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X) \to H^2(X^{\mathrm{an}}, \mathbb{Z}/n(1)) \to 0$$

if we use the statement that $H^2(X^{\mathrm{an}}, \mathcal{O}_X^*) = 0$ (as was proven above). Using that $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathrm{Pic}^0(X)$ with $\mathrm{Pic}^0(X)$ an n-divisible group, we can deduce Theorem 6.1 for $\Lambda = \mathbb{Z}/n$ because $\mathrm{Pic}^0(X)[n] \cong (\mathbb{Z}/n)^{2g}$ by the structure of Jacobian $\mathrm{Pic}^0(X) \cong \mathbb{C}^g/\mathbb{Z}^{2g}$ discussed before.

Let us now assume that X is a smooth, projective, connected curve over an algebraically closed field k (of arbitrary characteristic) and try to implement this last reasoning to calculate $H^*_{\text{ét}}(X, \mathbb{Z}/n)$.

The first task that we have to show then is the following:

Task 6.2. *For any scheme S the functors $U \in S_{\text{ét}} \mapsto \mathcal{O}_U(U)^*$, $U \mapsto \mathcal{O}_U(U)$ define étale sheaves denoted $\mathcal{O}^*$ and $\mathcal{O}$ on $S_{\text{ét}}$. More generally, for any scheme T over S the functor $U \mapsto \mathrm{Hom}_S(U, T)$ is an étale sheaf, cf. Corollary 7.14.*

Let us assume this for the rest of this section. We can then deduce the existence of the "mod n exponential sequence", which is more known as the Kummer sequence.

Lemma 6.3. *Let S be a scheme and $n \in Z$ with $n \in \mathcal{O}_S(S)^\times$. Then the sequence*

$$0 \to \mathbb{Z}/n(1) := \mu_n \to \mathcal{O}^* \xrightarrow{n} \mathcal{O}^* \to 0$$

is an exact sequence of étale sheaves on $S_{\text{ét}}$. Here, $\mu_n(U) := \{z \in \mathcal{O}^*(U) \mid z^n = 1\}$ is the sheaf of n-th roots of unity and $\mathbb{Z}/n(1) \cong \underline{\mathbb{Z}/n}$ if $\mathcal{O}_S(S)^\times$ contains a primitive n-th root of unity.

Proof. By definition of μ_n the sequence is left exact. Let $T \in S_{\text{ét}}$ and let $s \in \mathcal{O}^*(T)$ be section. Let $T = \bigcup\limits_{i \in I} T_i$ be an open cover by affines. Then $\{T_i \to T\}_{i \in I}$ is an étale cover and because étale covers can be composed, we may assume that $T = \mathrm{Spec}(A)$ is affine. Then

$$T' := \mathrm{Spec}(A[u]/(u^n - s))$$

is finite free, surjective and étale over T. Indeed, finite freeness and surjectivity is clear. For checking étaleness we use the Jacobian criterion. But the ideal $(nu^{n-1}, u^n - s) \subseteq A[u]/(u^n - s)$ contains the unit s because n is invertible in $\mathcal{O}_S(S)$ and hence in A. This implies exactness of the Kummer sequence. Let us now check that the natural map $\underline{\mathbb{Z}/n} \to \mathbb{Z}/n(1)$, $1 \mapsto \zeta_n$ is an isomorphism if $\zeta_n \in \mathcal{O}_S(S)$ is a primitive n-th root of unity, i.e., there exists a map $\mathbb{Z}[\zeta_n'] \to \mathcal{O}_S(S)$ with $\zeta_n' \in \mathbb{C}$ a primitive n-th root of unity. Now note that the étale sheaves $\underline{\mathbb{Z}/n}, \mathbb{Z}/n(1)$ on $S_{\text{ét}}$ are representable by the étale S-schemes $\coprod\limits_{\mathbb{Z}/n} S$, $\underline{\mathrm{Spec}}_{\mathcal{O}_S}(\mathcal{O}_S[u]/(u^n - 1))$. As a bijective étale morphism is an isomorphism, we may assume by base change that $S = \mathrm{Spec}(k)$ is an algebraically closed field. But then it is clear that $\underline{\mathbb{Z}/n} \to \mu_n$, $1 \to \zeta_n$ is an isomorphism of étale S-schemes, and hence of étale sheaves. □

The statement of Lemma 6.3 is wrong in general[35] and thus we see our first restriction: we have to assume that n is invertible in k. Let us assume this from now on, if not stated otherwise.

Given Lemma 6.3 we get a long exact sequence

$$0 \to \mathbb{Z}/n(1)(X) \to \mathcal{O}^*(X) \xrightarrow{n} \mathcal{O}^*(X) \to H^1_{\text{ét}}(X, \mathbb{Z}/n(1)) \to H^1_{\text{ét}}(X, \mathcal{O}^*) \xrightarrow{n} H^1_{\text{ét}}(X, \mathcal{O}^*) \xrightarrow{c_1}$$

[35] taking p-th roots in characteristic p produces inseparable maps.

$$\to H^2_{\text{ét}}(X,\mathbb{Z}/n(1)) \to H^2_{\text{ét}}(X,\mathcal{O}^*) \xrightarrow{n} H^2_{\text{ét}}(X,\mathcal{O}^*) \to H^3_{\text{ét}}(X,\mathbb{Z}/n(1) \to \ldots.$$

The first three terms are easy as $\mathcal{O}(X) \cong k$. To compute $H^1_{\text{ét}}(X,\mathbb{Z}/n(1)$ following the above strategy we find our next tasks, stated here in the form that we will prove:

Task 6.4. *For any scheme S there exists a natural isomorphism $H^1_{\text{ét}}(S,\mathcal{O}^*) \cong \mathrm{Pic}(S)$, cf. Corollary 7.21.*

Task 6.5. *For each curve Y over k the group*

$$H^i_{\text{ét}}(Y,\mathcal{O}^*)$$

vanishes for $i \geq 2$, cf. Theorem 9.5.

The last statement implies that $H^i_{\text{ét}}(X,\mathbb{Z}/n(1)) = 0$ for $i \geq 3$. Finally, we will also have to show (or rather quote, cf.) that

$$\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathrm{Pic}^0(X)$$

with $\mathrm{Pic}^0(X)$ an n-divisible group with n-torsion $\mathrm{Pic}^0(X)[n] \cong (\mathbb{Z}/n)^{2g}$, i.e., study Jacobians of curves.[36]

If we can settle all this, then we will have proven the following theorem.

Theorem 6.6. *Let k be an algebraically closed field and X a smooth, projective, connected curve of genus g over k. Let $n \in \mathbb{Z}$ with $n \in k^*$. Then*

$$H^i_{\text{ét}}(X,\mathbb{Z}/n(1)) \cong \begin{cases} \mathbb{Z}/n, & i = 0,2 \\ (\mathbb{Z}/n)^{2g}, & i = 1 \\ 0, & i \geq 3. \end{cases}$$

Let us analyze in more detail why the case $n = 0 \in k$ fails. The main tool is the following "additive Kummer sequence", most commonly called Artin-Schreier sequence.

Lemma 6.7. *Let p be a prime and let S be a scheme over $\mathbb{F}_p$. Then the sequence*

$$0 \to \underline{\mathbb{F}_p} \to \mathcal{O} \xrightarrow{x \mapsto x^p - x} \mathcal{O} \to 0$$

of étale sheaves on $S_{\text{ét}}$ is exact.

Proof. By Task 6.2 $\mathcal{O}$ is an étale sheaf. Granting this the proof of surjectivity proceeds exactly as in Lemma 6.3 by using that for any $U = \mathrm{Spec}(A) \to S$ étale and $a \in \mathcal{O}(U) = A$ the morphism $\mathrm{Spec}(A[T]/T^p - T - a) \to U$ is (finite) étale (as $p = 0$ the derivative of $T^p - T - a$ is -1). In fact, this argument shows that the map $A^1_{\mathbb{F}_p} \to \mathbb{A}^1_{\mathbb{F}_p}$, $x \mapsto x^p - x$ is finite étale. In particular, its kernel $K \cong \mathrm{Spec}(\mathbb{F}[T]/T^p - T)$ is a finite étale $\mathbb{F}_p$-scheme and in fact $K \cong \underline{\mathbb{F}_p}$ because $T^p - T = \prod_{a \in \mathbb{F}_p} (T - a)$. Restricting to $S_{\text{ét}}$ yields the claim. □

Thus, if S is any scheme over $\mathbb{F}_p$, then there exists a long exact sequence

$$0 \to \underline{\mathbb{F}_p}(S) \to \mathcal{O}(S) \xrightarrow{x \mapsto x^p - x} \mathcal{O}(S) \to H^1_{\text{ét}}(S,\mathbb{F}_p) \to H^1_{\text{ét}}(S,\mathcal{O}) \to H^1_{\text{ét}}(S,\mathcal{O}) \to \ldots$$

This sequence can be made more concrete by the following statement that we will prove.

Task 6.8. *For any scheme S and any quasi-coherent $\mathcal{O}_S$-module $\mathcal{M}$ the functor $(U \xrightarrow{f} S) \mapsto f^*\mathcal{M}(U)$ is an étale sheaf on $S_{\text{ét}}$, again called $\mathcal{M}$, and for $i \geq 0$ there exists a natural isomorphism $H^i_{\text{ét}}(S,\mathcal{M}) \cong H^i_{\text{Zar}}(S,\mathcal{M})$, where the latter denotes the usual sheaf cohomology of $\mathcal{M}$ for the Zariski topology on S.*

In particular, for S over $\mathbb{F}_p$ we can conclude that $H^i_{\text{ét}}(S,\mathbb{F}_p) = 0$ for $i > \dim(S) + 1$. While this does not provide an immediate contradiction in the case that X is a curve over an algebraically closed field k of characteristic p[37], it implies that $H^*_{\text{ét}}(\mathbb{P}^n_k,\mathbb{F}_p)$ cannot have the expected shape if $n \geq 2$.

We will now start to prove the several tasks that we highlighted.

[36] As the seminar last term was on this topic we will be short on this point and refer to [Stacks, Tag 0B92] and [22] (for the statements on abelian varieties).

[37] If $S \to \mathrm{Spec}(k)$ is proper and $k/\mathbb{F}_p$ an algebraically closed field, then actually $H^{\dim(S)+1}(S,\mathbb{F}_p) = 0$ and thus even the $\underline{\mathbb{F}_p}$-cohomology of proper, smooth curves cannot behave as expected, cf. [Stacks, Tag 0A3L].

7. Faithfully flat descent of quasi-coherent sheaves

In this section we prove faithfully flat descent for quasi-coherent sheaves on schemes.

7.1. **Faithfully flat descent for quasi-coherent sheaves.** Let us motivate the question on descent by the following (very simple) example.

Example 7.2. Let $f\colon Y \to X$ be a surjection of sets. Let $g\colon X' \to X$ be any morphism. Set $h\colon Y' := Y \times_X X' \to Y$. If $y \in Y$ with image $x \in X$. Then there exists a canonical isomorphism

$$\alpha_y\colon h^{-1}(y) \cong g^{-1}(x)$$

induced by the projection $Y' \to X'$. If $y_1, y_2 \in Y$ are two elements with same image $x \in X$ there exists therefore the canonical identification

$$\alpha_{y_1,y_2} := \alpha_{y_2}^{-1} \circ \alpha_{y_1}\colon h^{-1}(y_1) \to h^{-1}(y_2)$$

and if $y_1, y_2, y_3 \in Y$ have image x, then the cocycle condition

$$\alpha_{y_2,y_3} \circ \alpha_{y_1,y_2} = \alpha_{y_1,y_3}$$

holds. From these data we can reconstruct X' and the morphism $g\colon X' \to X$ completely. Namely, given $h\colon Y' \to Y$ and the isomorphisms α_{y_1,y_2} for $y_1, y_2 \in Y$ with $f(y_1) = f(y_2)$ satisfying the cocycle condition, then define

$$Z := Y'/\sim$$

with $y_1' \sim y_2'$ if $h(f(y_1')) = h(f(y_2'))$ and $\alpha_{f(y_1'),f(y_2')}(y_1') = y_2'$. The cocycle condition is necessary to see that $\sim$ is actually an equivalence relation. The data of all these α_{y_1,y_2} can conveniently be packaged as follows: Let $p_1, p_2\colon Y \times_X Y \to Y$ be the two projections. Then we get the isomorphism

$$\alpha\colon p_1^*(Y') := (Y \times_X Y) \times_{Y,p_1} Y' \cong p_2^*(Y') := (Y \times_X Y) \times_{Y,p_2} Y'$$

over $Y \times_X Y$ defined by

$$\alpha(((y_1, y_2), y') := ((y_1, y_2), \alpha_{y_1,y_2}(y'))$$

(note that $((y_1, y_2), y') \in p_1^*(Y')$ implies $y_1 = h(y')$. The cocycle condition translates into the condition that

$$p_{23}^*(\alpha) \circ p_{12}^*(\alpha) = p_{13}^*(\alpha) \tag{2}$$

as morphisms $p_{12}^* p_1^*(Y') \to p_{13}^* p_2^*(Y')$, where $p_{ij}\colon Y \times_X Y \times_X Y \to Y \times_X Y$ denote the projection in the i,j-factor. We arrive at the equivalence between the category (Sets$/X$) of sets over X and the category of "descent data" for f, i.e., the category of pairs $(h\colon Y' \to Y, \alpha\colon p_1^*(Y') \cong p_2^*(Y'))$ such that α satisfies the cocycle condition over $Y \times_X Y \times_X Y$, and morphisms, which respect the descent data.

If $f\colon Y \to X$ is a surjective map of topological spaces, which is universally a quotient map (e.g., surjective and universally open or closed), then we arrive at similar equivalence for topological spaces.

Let $f\colon Y \to X$ be a morphism of schemes. We make the convention that

$$p_i\colon Y \times_X Y \to Y,\ p_{ij}\colon Y \times_X Y \times_X Y \to Y \times_X Y$$

denote the resp. projection on the i-th or (i,j)-th factor from the (scheme-theoretic) fiber product. Motivated by Example 7.2 we make the following definition.

Definition 7.3.
(1) A descent datum on a Y-scheme $Y' \to Y$ is an isomorphism

$$\alpha\colon p_1^* Y' := (Y \times_X Y) \times_{Y,p_1} Y' \to p_2 Y' := (Y \times_X Y) \times_{Y,p_2} Y'$$

satisfying the cocycle condition Equation (2).

(2) A morphism $g\colon (Y', \alpha) \to (Y'', \beta)$ of descent data for schemes is a morphism $g\colon Y' \to Y''$ of Y-schemes, such that the diagram

$$\begin{array}{ccc} p_1^*(Y') & \xrightarrow{p_1^*(g)} & p_1^*(Y'') \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ p_2^*(Y') & \xrightarrow{p_2^*(g)} & p_2^*(Y'') \end{array}$$

commutes. We denote by $\mathrm{Desc}_{Y/X}(\mathrm{Sch})$ the category of descent data for schemes.

(3) Similarly, we define a descent datum on a quasi-coherent sheaf $\mathcal{M}$ on Y as an isomorphism $\alpha\colon p_1^*(\mathcal{M}) \to p_2^*(\mathcal{M})$ satisfying the cocycle condition.

(4) Similarly we define a morphism for descent data for quasi-coherent sheaves. We denote by $\mathrm{Desc}_{Y/X}(\mathrm{QCoh})$ the category of descent data for quasi-coherent sheaves.

(5) We call a descent datum (Y', α) effective if there exists an X-scheme X' and an isomorphism $\varphi\colon Y' \to f^*(X) := Y \times_X X'$ such that α is the composition
$$p_1^*(Y') \xrightarrow{p_1^*(\varphi)} p_1^*(f^*(X')) \cong p_2^*(f^*(X')) \xrightarrow{p_2^*(\varphi^{-1}} p_2^*(Y'),$$
where the middle isomorphism comes from the equality $f \circ p_1 = f \circ p_2$. Similarly, we define effectiveness for descent data of modules.

Let us discuss the following examples.

Example 7.4. Assume that X is a scheme and $X = \bigcup\limits_{i \in I} U_i$ is an open cover. Set $f\colon Y := \coprod\limits_{i \in I} U_i \to X$ as the natural surjection. Then $Y \times_X Y \cong \coprod\limits_{i,j} U_i \cap U_j$ and a descent datum for f identfies with a collection of U_i-schemes X_i and isomorphisms
$$\alpha_{i,j}\colon X_{i|U_i \cap U_j} \cong X_{j|U_i \cap U_j}$$
satisfying the cocycle condition. In particular, we see that every descent datum is effective in this case as this reduces to glueing the X_i as in Algebraic Geometry I. Similarly, each descent data for quasi-coherent sheaves (even any $\mathrm{Mod}_{\mathcal{O}_Y}$) is effective by glueing sheaves.

Much more generally, we have the following result.

Theorem 7.5. *Assume that $f\colon Y \to X$ is faithfully flat and quasi-compact. Then the functor $\Phi_{Y/X}\colon \mathrm{QCoh}(X) \to \mathrm{Desc}_{Y/X}(\mathrm{QCoh})$ is an equivalence.*

Proof. Using Example 7.4 the general case of Theorem 7.5 reduces to the case that $Y = \mathrm{Spec}(B) \to X = \mathrm{Spec}(A)$ are affine. More details on the reduction can be found in [Stacks, Tag 023T]. Then the proof follows from Lemma 7.10 and Lemma 7.11 below. □

Let us note the following general observation about descent data.

Lemma 7.6. *Let $\alpha\colon p_1^*\mathcal{N} \cong p_2^*\mathcal{N}$ be a descent datum on $\mathcal{N} \in \mathrm{QCoh}(Y)$, and let $\Delta\colon Y \to Y \times_X Y$ be the diagonal. Then*
$$\Delta^*(\alpha)\colon \mathcal{N} \cong \Delta^* p_1^*\mathcal{N} \to \Delta^* p_2^*\mathcal{N} \cong \mathcal{N}$$
is the identity. Conversely, assume that $\alpha\colon p_1^*\mathcal{N} \to p_2^*\mathcal{N}$ is a morphism satisfying the cocycle condition such that $\Delta^*(\alpha)$ is the identity. Then α is an isomorphism, and thus defines a descent datum.

Proof. As $\Delta^*(\alpha)$ is an isomorphism, it suffices to show that
$$\Delta^*(\alpha) \circ \Delta^*(\alpha) = \Delta^*(\alpha).$$
But this follows from restricting the cocycle condition
$$p_{23}^*(\alpha) \circ p_{12}^*(\alpha) = p_{13}^*(\alpha),$$
along the diagonal embedding $Y \to Y \times_X Y \times_X Y$.

Now let us prove the second statement. Let $\sigma\colon Y \times_X Y \to Y \times_X Y,\ (y_1, y_2) \mapsto (y_2, y_1)$ be the flip. Then we claim that
$$\sigma^*(\alpha)\colon p_2^*\mathcal{N} \cong \sigma^* p_1^*\mathcal{N} \to \sigma^* p_2^*\mathcal{N} \cong p_1^*\mathcal{N}$$
is an inverse to α. The equality $\alpha \circ \sigma^*(\alpha) = \mathrm{Id}$ follows by restricting the cocycle condition
$$p_{23}^*(\alpha) \circ p_{12}^*(\alpha) \cong p_{13}^\circ(\alpha)$$
on $Y \times_X Y \times_X Y$ along the morphism
$$\iota\colon Y \times_X Y \to Y \times_X Y \times_X Y,\ (y_1, y_2) \mapsto (y_1, y_2, y_1)$$
and using that $p_{13} \circ i = \Delta \circ p_1$. The argument for $\sigma^*(\alpha) \circ \alpha = \mathrm{Id}$ follows similarly using the embedding $(y_1, y_2) \mapsto (y_2, y_1, y_2)$. □

Let $\varphi\colon A \to B$ be a morphism of rings. We set $\mathrm{Desc}_{B/A} := \mathrm{Desc}_{Y/X}(\mathrm{QCoh})$ and discuss now descent for quasi-coherent sheaves (or equivalently modules). If $M \in \mathrm{Mod}_A$, then $B \otimes_A M$ admits the canonical descent datum
$$\alpha_{\mathrm{can}}\colon (B \otimes_A B) \otimes_{B,\iota_1} (B \otimes_A M) \cong B \otimes_A B \otimes_A M \cong (B \otimes_A B) \otimes_{B,\iota_2} (B \otimes_A M),$$
where $\iota_1, \iota_2\colon B \to B \otimes_A B$ denote the two inclusions $b \mapsto b \otimes 1$ resp. $b \mapsto 1 \otimes b$. For $N \in \mathrm{Mod}_B$ let us identify
$$(B \otimes_A B) \otimes_{B,\iota_1} N \cong N \otimes_A B,\ (b_1 \otimes b_2) \otimes n \mapsto b_1 n \otimes b_2,$$

which intertwines the $B \otimes_A B$-action on the left with the action $(b_1 \otimes b_2)(n \otimes c) = b_1 n \otimes b_2 c$ on the right. Similarly, we identify

$$(B \otimes_A B) \otimes_{B,\iota_2} N \cong B \otimes_A N,\ (b_1 \otimes b_2) \otimes n \mapsto b_1 \otimes b_2 n,$$

which intertwines the $B \otimes_A B$-action on the left with the action $(b_1 \otimes b_2)(c \otimes n) = b_1 c \otimes b_2 n$. Thus, we can identify a descent datum α with a $B \otimes_A B$-linear isomorphism

$$\alpha\colon N \otimes_A B \to B \otimes_A N$$

satisfying the cocycle condition. We define

$$\alpha_0\colon N \to B \otimes_A N,\ n \mapsto \alpha(n \otimes 1),$$

which is a B-linear map if B acts via ι_1 on $B \otimes_A N$, i.e., via the factor B. The following lemma gives a description of descent data in terms of α_0.

Lemma 7.7. (1) *The diagram*

$$\begin{array}{ccc} N & \xrightarrow{\alpha_0} & B \otimes_A N \\ {\scriptstyle\alpha_0}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_B \otimes \alpha_0} \\ B \otimes_A N & \xrightarrow{\iota_1 \otimes \mathrm{Id}_N} & B \otimes_A B \otimes_A N \end{array}$$

commutes (this encodes "coassociatitiy" of α_0). Moreover, the composition

$$N \xrightarrow{\alpha_0} B \otimes_A N \xrightarrow{b \otimes n \mapsto bn} N$$

is the identity (this encodes "counitality").

(2) *Conversely, given an $N \in \mathrm{Mod}_B$ and a map $\alpha_0\colon N \to B \otimes_A N$, which is linear over $\iota_1\colon B \to B \otimes_A B$, such that the above diagram commutes and the above composition is the identity, then*

$$(N, \alpha\colon N \otimes_A B \xrightarrow{n \otimes b \mapsto (1 \otimes b)\cdot \alpha_0(n)} B \otimes_A N)$$

is a descent datum on N.

Clearly, we can also identify morphisms of descent data with morphisms of B-modules, which preserve the respective α_0's.

Proof. We check the first point. The cocycle condition implies that the diagram

$$\begin{array}{ccc} N \otimes_A B \otimes_A B & \xrightarrow{\alpha \otimes \mathrm{Id}_B} & B \otimes_A N \otimes_A B \\ & \searrow^{\beta} & \downarrow{\scriptstyle \mathrm{Id}_B \otimes \alpha} \\ & & B \otimes_A B \otimes_A N \end{array}$$

commutes, where $\beta(n \otimes b \otimes c) = \sum_i c_i \otimes b \otimes n_i$ if $\alpha(n \otimes c) = \sum_i c_i \otimes n_i$. Evaluating at $n \otimes 1 \otimes 1$ yields the commutativity for diagram with α_0. The second statement is an algebraic reformulation of the first assertion in Lemma 7.6. Let us check the converse and assume that $\alpha_0\colon N \to B \otimes_A N$ is linear over ι_1, or equivalently it is linear if B acts on $B \otimes_A N$ via the first factor, and such that α_0 satisfies coassociativity and counitality. Then α satisfies the condition of the second assertion in Lemma 7.6 and we can conclude that α is a descent datum. □

Remark 7.8. In categorical terms Lemma 7.7 says that descent data identify with the comodules for the comonad

$$N \mapsto B \otimes_A |N|$$

on Mod_B coming from the adjunction between $B \otimes_A (-)$ and the forgetful functor $|-|\colon \mathrm{Mod}_B \to \mathrm{Mod}_A$. From this perspective Theorem 7.5 becomes an application of the Barr-Beck theorem, cf. [7, p. 4.2]. We will however give a direct argument.

In the following we will identify descent data (N, α) with pairs $(N, \alpha_0\colon N \to B \otimes_A N)$ as in Lemma 7.7. In these terms, the canonical descent datum on $B \otimes_A M$ for $M \in \mathrm{Mod}_A$ is the map

$$\alpha_{0,\mathrm{can}}\colon B \otimes_A M \to B \otimes_A B \otimes_A M,\ b \otimes m \mapsto b \otimes 1 \otimes m.$$

Lemma 7.9. *The natural functor $\Phi\colon \mathrm{Mod}_A \to \mathrm{Desc}_{B/A}$, $M \mapsto (M, \alpha_{0,\mathrm{can}})$ has a right adjoint $\Psi\colon \mathrm{Desc}_{B/A} \to \mathrm{Mod}_A$ is defined by the equalizer*

$$(N, \alpha_0) \mapsto \mathrm{eq}(N \rightrightarrows B \otimes_A N) = \{n \in N \mid \alpha_0(n) = 1 \otimes n\}$$

for the morphism $\alpha_0\colon N \to B \otimes_A N$ and the morphism $N \xrightarrow{n \mapsto 1 \otimes n} B \otimes_A N$ (which is linear over ι_2, but not over ι_1). Moreover, Φ, Ψ commute with base change along a flat morphism $A \to A'$.

Proof. The compatibility with base change along flat morphisms is clear. Let us check the adjunction. Take $M \in \mathrm{Mod}_A$ and $(N, \alpha_0) \in \mathrm{Desc}_{B/A}$. Then

$$\begin{aligned} & \mathrm{Hom}_{\mathrm{Desc}_{B/A}}((B \otimes_A M, \alpha_{0,\mathrm{can}}), (N, \alpha_0)) \\ = \ & \{f \colon M \to N \ A\text{-linear} \mid 1 \otimes f(m) = \alpha_0(f(m)) \text{ for all } m \in M\} \\ = \ & \mathrm{Hom}_{\mathrm{Mod}_A}(M, \Psi(N, \alpha_0)) \end{aligned}$$

using that $\mathrm{Hom}_{\mathrm{Mod}_B}(B \otimes_A M, N) \cong \mathrm{Hom}_{\mathrm{Mod}_A}(M, N)$. □

Lemma 7.10. *Assume $\varphi \colon A \to B$ is faithfully flat. For any $M \in \mathrm{Mod}_A$ the sequence*

$$0 \to M \to B \otimes_A M \xrightarrow{\beta} B \otimes_A B \otimes_A M,$$

is exact, where β is the map $b \otimes m \mapsto b \otimes 1 \otimes m - 1 \otimes b \otimes m$. In particular, the unit $M \to \Psi(\Phi(M)$ is an isomorphism, and $\Phi \colon \mathrm{Mod}_A \to \mathrm{Desc}_{B/A}$ is fully faithful.

Proof. We may check the statement after some faithfully flat base change $A \to A'$. Setting $A' = B$ reduces us to the case that there exists some section $\sigma \colon A \to B$, i.e, $\varphi \circ \sigma = \mathrm{Id}_A$. This already implies injectivity of

$$M \to B \otimes_A M, \ m \mapsto 1 \otimes m.$$

Let $\sum\limits_i b_i \otimes m_i \in B \otimes_A M$ such that

$$\sum_i 1 \otimes b_i \otimes m_i = \sum_i b_i \otimes 1 \otimes m_i.$$

Applying the map $\mathrm{Id}_B \otimes \sigma \otimes \mathrm{Id}_M \colon B \otimes_A B \otimes_A M \to B \otimes_A A \otimes_A M \cong B \otimes_A M$ yields

$$\sum_i 1 \otimes \sigma(b_i) m_i = \sum_i b_i \otimes m_i,$$

and thus exactness in the middle. The remaining statement follow directly. □

Lemma 7.11. *Assume that $\varphi \colon A \to B$ is faithfully flat. Let $(N, \alpha_0 \colon N \to B \otimes_A N) \in \mathrm{Desc}_{B/A}$ and $M := \Psi(N, \alpha_0) = \mathrm{eq}(N \rightrightarrows B \otimes_A N) \in \mathrm{Mod}_A$. Then the natural map*

$$B \otimes_A M \to N$$

is an isomorphism. In particular, the counit $\Phi \circ \Psi(N, \alpha_0) \to (N, \alpha_0)$ is an isomorphism and Φ, Ψ are equivalences.

Proof. As in Lemma 7.10 we may base change with $A' = B$ and assume there exists a section $\sigma \colon B \to A$ of φ. Let $\sum\limits_i b_i \otimes m_i \in B \otimes_A M$ such that $\sum\limits_i b_i m_i = 0 \in N$. Now apply the B-linear map $\alpha_0 \colon N \to B \otimes_A N$ to $\sum\limits_i b_i m_i$. This shows

$$0 = \alpha_0(\sum_i b_i m_i) = \sum_i b_i \alpha_0(m_i) = \sum_i b_i \otimes m_i$$

by definition of M. This show injectivity. Now take $n \in N$ and write $\alpha_0(n) = \sum\limits_i b_i \otimes n_i \in B \otimes_A N$. Counitality of the coaction α_0, cf. Lemma 7.7, implies that $n = \sum\limits_i b_i n_i$. Write

$$\alpha_0(n_i) = \sum_j b_{i,j} \otimes n_{i,j}.$$

By coassociativity (aka the cocycle condition) we have

$$\sum_{i,,j} b_i \otimes b_{i,j} \otimes n_{i,j} = \sum_{i,j} b_i \otimes 1 \otimes n_i \in B \otimes_A B \otimes_A N.$$

Now apply $\sigma \otimes \mathrm{Id}_B \otimes \mathrm{Id}_N$ to this equality. This yields

$$\sum_{i,,j} \sigma(b_i) b_{i,j} \otimes n_{i,j} = \sum_{i,j} \sigma(b_i) \otimes n_i = \sum_i 1 \otimes \sigma(b_i) n_i$$

as $\sigma(b_i) \in A$. Set $n_\sigma := \sum\limits_i \sigma(b_i) n_i = \sigma \otimes \mathrm{Id}_N \circ \alpha_0(n)$. As α_0 is B-linear, we can conclude from the above equality that

$$\alpha_0(n_\sigma) = 1 \otimes n_\sigma,$$

i.e., $n_\sigma \in M$. Now apply $\mathrm{Id}_B \otimes \sigma \otimes \mathrm{Id}_N$ instead. This yields that

$$\sum_{i,j} b_i \otimes \sigma(b_{i,j}) n_{i,j} = \sum_i b_i \otimes (n_i)_\sigma = \sum_i b_i \otimes n_i,$$

and thus
$$n = \sum_i b_i n_i = \sum_i b_i (n_i)_\sigma$$
lies in the image of $B \otimes_A M \to N$ as each $(n_i)_\sigma \in M$ as was shown above.. □

7.12. **Consequences.** Theorem 7.5 has many interesting corollaries.

Corollary 7.13. *Assume that $f\colon Y \to X$ is faithfully flat and quasi-compact.*

(1) *The functor* $\mathrm{Sch}/X \to \mathrm{Desc}_{Y/X}(\mathrm{Sch})$ *is fully faithful.*
(2) *Each descent datum* $(Y', \alpha) \in \mathrm{Desc}_{Y/X}(\mathrm{Sch})$ *with* $g\colon Y' \to Y$ *affine, is effective.*

Proof. Let $Z \to X, Z' \to X$ be two morphisms of schemes. As we can construct morphisms of schemes locally on affine opens, we can reduce to the case that Z, Z', X, Y are affine. But then every scheme is determined by its ring of global sections, and it follows easy from Theorem 7.5 that the base change $B \otimes_A (-)$ induces an equivalence between the category of A-algebras and the category of descent data (C, α_0) such that C is a B-algebra and α_0 a morphism of B-algebras. This settles the fully faithfulness and also the effective descent for affine morphisms. □

In general, descent data for schemes are not effective, even if g is projective ([Stacks, Tag 08KE]).

Corollary 7.14. *Let X be a scheme, $\mathcal{M} \in \mathrm{QCoh}(X)$ and $Y \to X$ a morphism of schemes. The functors*

(1) $S \in \mathrm{Sch}/X \mapsto \mathrm{Hom}_X(S, Y)$
(2) $(g\colon S \to X) \in \mathrm{Sch}/X \mapsto \mathcal{M}_S(S)$ *with* $\mathcal{M}_S := g^*\mathcal{M}$

are fpqc-sheaves, and thus a fortiori étale sheaves.

Proof. By the definition of the fpqc-topology it follows that it suffices to check that for any faithfully flat map $S' := \mathrm{Spec}(B) \to S := \mathrm{Spec}(A)$ with
$$S'' := S' \times_S S' = \mathrm{Spec}(B \otimes_A B)$$
the sequence
$$Y(S) \to Y(S') \rightrightarrows Y(S'')$$
is exact. Glueing morphisms, we can further reduce to the case that $Y = \mathrm{Spec}(C), X = \mathrm{Spec}(R)$ are affine. But then this sequence identifies with
$$\mathrm{Hom}_R(C, A) \to \mathrm{Hom}_R(C, B) \rightrightarrows \mathrm{Hom}_R(C, B \otimes_A B)$$
and exactness follows from Lemma 7.10 (set $M = B$). Again we can reduce to the case that $X, S = \mathrm{Spec}(A), S' = \mathrm{Spec}(B)$ are affine. Replacing $\mathcal{M}$ by $\mathcal{M}_S$ we may assume $X = S$. Then $\mathcal{M} = \widetilde{M}$ for some $M \in \mathrm{Mod}_A$. The sequence
$$\mathcal{M}_S(S) \to \mathcal{M}_{S'}(S') \rightrightarrows \mathcal{M}_{S' \times_S S'}(S' \times_S S')$$
identifies with
$$M \to M \otimes_A B \rightrightarrows M \otimes_A (B \otimes_A B)$$
and exactness follows from Lemma 7.10. □

Remark 7.15. Let $\mathfrak{X}$ be any topos and let $\mathcal{O}_\mathfrak{X}$ be a sheaf of rings on $\mathfrak{X}$ aka a ring object. Then the definitions in Definition 3.8 apply (almost) literally to the ringed topos $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$. Namely, an $\mathcal{O}_\mathfrak{X}$-module $\mathcal{M}$ is quasi-coherent if there exists a covering $\{X_i \to *\}_{i \in I}$ of the terminal object $* \in \mathfrak{X}$ such that $\mathcal{M}_{|X_i}$ is the cokernel of an $\mathcal{O}_{\mathfrak{X}|X_i}$-linear map $\mathcal{O}_{\mathfrak{X}|X_i}^{J_1} \to \mathcal{O}_{\mathfrak{X}|X_i}^{J_2}$ on $\mathfrak{X}/X_i$. Note that pullback along morphisms of ringed topoi preserves quasi-coherent modules. We denote by $\mathrm{QCoh}(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ the category of quasi-coherent $\mathcal{O}_\mathfrak{X}$-modules.

Lemma 7.16. *Let X be scheme and let $\mathcal{O}_{X_{\text{ét}}}$ be the étale sheaf $S \mapsto \mathcal{O}_S(S)$ on $X_{\text{ét}}$. Let $\varepsilon\colon \widetilde{X_{\text{ét}}} \to \widetilde{X_{\text{Zar}}}$ be the natural morphism of ringed topoi. Then the $\varepsilon^*\colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_{X_{\text{ét}}}}$ restricts to an equivalence*
$$\varepsilon^*\colon \mathrm{QCoh}(X) \cong \mathrm{QCoh}(X_{\mathrm{Zar}}, \mathcal{O}_{X_{\mathrm{Zar}}}) \to \mathrm{QCoh}(\widetilde{X_{\text{ét}}}, \mathcal{O}_{X_{\text{ét}}}),\ \mathcal{M} \mapsto [(S \to X) \mapsto \mathcal{M}_S(S)].$$

The same argument applies if $X_{\text{ét}}$ is replaced by X_{fppf} of X_{fpqc}.[38]

[38] To avoid set theoretic issues one can fix some cut-off cardinal and then observe that the result is independent of that cut-off cardinal.

Proof. Note that by Corollary 7.14 $\varepsilon^*\mathcal{M} = \mathcal{O}_{X_{\text{ét}}} \otimes_{\varepsilon^{-1}\mathcal{O}_X} \mathcal{M}$ is given by sending $(S \to X)$ to $\mathcal{M}_S(S)$. We claim that ε_* provides a quasi-inverse to ε^*. Let $\mathcal{M} \in \mathrm{QCoh}(X)$ then the natural map

$$\mathcal{M} \to \varepsilon_*(\varepsilon^*\mathcal{M})$$

is an isomorphism because for $U \subseteq X$ open we have

$$\varepsilon_*(\varepsilon^*\mathcal{M})(U) = \varepsilon^*\mathcal{M}(U) = \mathcal{M}_{|U}(U) = \mathcal{M}(U).$$

Let $\mathcal{N} \in \mathrm{QCoh}(\widetilde{X_{\text{ét}}}, \mathcal{O}_{\mathfrak{X}})$. We claim that $\varepsilon_*\mathcal{N}$ is a quasi-coherent $\mathcal{O}_X$-modules and that the morphism

$$\Phi\colon \varepsilon^*\varepsilon_*\mathcal{N} \to \mathcal{N}$$

is an isomorphism. These claims are local on X (using Corollary 5.40). Hence we may assume that X is affine. Let $\{X_i \to X\}_{i\in I}$ be a cover in $X_{\text{ét}}$ such that $\mathcal{N}_{|X_i}$ is the cokernel of $\alpha_i\colon \mathcal{O}^{J_1}_{X_{i,\text{ét}}} \to \mathcal{O}^{J_2}_{X_{i,\text{ét}}}$. By quasi-compactness of X we may assume that I is finite, cf. Theorem 5.24. Taking a disjoint union we may even assume that $I = \{0\}$ is a singleton and that X_0 is affine. Set $N_0 := \mathrm{coker}(\alpha_0\colon \mathcal{O}^{J_1}_{X_{0,\text{ét}}}(X_0) \to \mathcal{O}^{J_2}_{X_{0,\text{ét}}}(X_0))$. By the proven fully faithfulness (applied to X_0) and right-exactness of ε_0^* for the morphism $\varepsilon_0\colon \widetilde{X_{0,\text{ét}}} \to \widetilde{X_{0,\text{Zar}}}$ we know that

$$\mathcal{N}_{|X_0} \cong \varepsilon_0^*\widetilde{N_0}.$$

Set $\mathcal{B}$ as the full subcategory of $Y \in X_{\text{ét}}$ affine such that there exists a morphism $Y \to X_0$ in $X_{\text{ét}}$. For any morphism $f\colon Z \to Y$ in $\mathcal{B}$ the natural map

$$\mathcal{O}_Z(Z) \otimes_{\mathcal{O}_Y(Y)} \mathcal{N}(Y) \to \mathcal{N}(Z) \tag{3}$$

is then an isomorphism (by Corollary 7.14 and the construction of ε_0^*). Let now $Y \in X_{\text{ét}}$ be affine and set $Y_0 := X_0 \times_X Y$. Then $\{Y_0 \to Y\}$ is a covering and thus the sequence

$$0 \to \mathcal{N}(Y) \to \mathcal{N}(Y_0) \xrightarrow{p_1^*-p_2^*} \mathcal{N}(Y_0 \times_Y Y_0)$$

is exact, where $p_1, p_2\colon Y_0 \times_Y Y_0 \to Y_0$ are the two projections. Now,

$$p_1^*\mathcal{N}(Y_0) \cong \mathcal{N}(Y_0 \times_Y Y_0) \cong p_2^*\mathcal{N}(Y_0)$$

defines a descent datum on $\mathcal{N}(Y_0)$. Let N_Y be its descent to an $\mathcal{O}_Y(Y)$-module by Theorem 7.5. Then the above exact sequence and Lemma 7.10 yield a natural isomorphism $\mathcal{N}(Y) \cong N_Y$ and $\mathcal{N}(Y_0) \cong \mathcal{O}_{Y_0}(Y_0) \otimes_{\mathcal{O}_Y(Y)} \mathcal{N}(Y)$. Using (Equation (3)) one checks that for any morphism $Y' \to Y$ of affine schemes in $X_{\text{ét}}$ the map

$$\mathcal{O}_{Y'}(Y') \otimes_{\mathcal{O}_Y(Y)} \mathcal{N}(Y) \cong \mathcal{O}_{Y'}(Y') \otimes_{\mathcal{O}_Y(Y)} N_Y \to N_{Y'} \cong \mathcal{N}(Y')$$

is an isomorphism. This then implies that $\varepsilon_*\mathcal{N}$ is quasi-coherent, and that Φ is an isomorphism. □

We can now also calculate étale cohomology (even fpqc-cohomolgy) of quasi-coherent sheaves.

Lemma 7.17. *Let X be a scheme, let $\mathcal{M} \in \mathrm{QCoh}(X)$ and $\varepsilon = \varepsilon_X\colon \widetilde{X_{\text{ét}}} \to \widetilde{X_{\text{Zar}}}$ be natural morphism (of ringed topoi). Then the natural map*

$$\Phi\colon \mathcal{M} \to R\varepsilon_*(\varepsilon^*\mathcal{M})$$

is an isomorphism. In particular,

$$R\Gamma(X_{\text{Zar}}, \mathcal{M}) \cong R\Gamma(X_{\text{Zar}}, R\varepsilon_*(\varepsilon^*\mathcal{M})) \cong R\Gamma(X_{\text{ét}}, \varepsilon^*\mathcal{M}).$$

The same proof shows the same for $X_{\text{ét}}$ replaced by X_{fpqc} or X_{fppf}.

Proof. The statement that Φ is an isomorphism is local on X (by Corollary 5.40). Hence, we may assume that X is affine. We need to show that

$$H^i_{\text{ét}}(X, \varepsilon^*\mathcal{M}) = 0$$

for $i > 0$. As an extension of a quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-modules on $X_{\text{ét}}$ with quotient $\mathcal{O}_{X_{\text{ét}}}$ are again quasi-coherent we can conclude the case $i = 1$ by Lemma 7.16 from the case of the Zariski topoloy. Now use induction on $i \geq 2$. Then Corollary 5.40 implies that for any map $f\colon Y \to X$ of affine schemes with morphism $f_{\text{ét}}\colon \widetilde{Y_{\text{ét}}} \to \widetilde{X_{\text{ét}}}$ on étale topoi, and any $\mathcal{N} \in \mathrm{QCoh}(Y)$ we have

$$R^j f_{\text{ét},*}(\varepsilon_Y^*\mathcal{N}) = 0$$

for $j = 1, \ldots, i-1$. Moreover, one checks that $f_{\text{ét},*}(\varepsilon_Y^*(\mathcal{N})) \cong \varepsilon_X^* f_*(\mathcal{N})$ by evaluating on affines in $X_{\text{ét}}$. Now, the usual strategy applies, cf. Lemma 4.20. □

7.18. **Consequences for torsors.** We start with the following generalization of Lemma 4.39.

Lemma 7.19. *Let $\mathfrak{X}$ be a (small) topos and $\mathcal{G}$ a sheaf of abelian groups on $\mathfrak{X}$ (aka abelian group object). Then there exists a natural isomorphism*

$$H^1(\mathfrak{X}, \mathcal{G}) \cong \{\mathcal{G} - \textit{torsors } \mathcal{P}\}/\simeq.$$

Here, a $\mathcal{G}$-torsor is a sheaf (of sets) $\mathcal{P}$ on $\mathfrak{X}$ (aka object) with a right $\mathcal{G}$-action such that $\mathcal{P} \to *$ is an epimorphisms, i.e., a covering in $\mathfrak{X}$, and such that natural map

$$\mathcal{P} \times \mathcal{G} \to \mathcal{P} \times \mathcal{P},\ (p, g) \mapsto (p, pg)$$

is an isomorphism.

Proof. Both sides can be identified with isomorphism class of extensions

$$0 \to \mathcal{G} \to \mathcal{E} \to \underline{\mathbb{Z}} \to 0$$

in $\mathrm{Sh}_{\mathrm{Ab}}(\mathfrak{X})$ as

$$H^1(\mathfrak{X}, \mathcal{G}) \cong \mathrm{Ext}^1_{\mathrm{Sh}_{\mathrm{Ab}}(\mathfrak{X})}(\underline{\mathbb{Z}}, \mathcal{G}).$$

This finishes the proof. □

Definition 7.20. For $\mathcal{G}$ non-abelian, we define $H^1(\mathfrak{X}, \mathcal{G})$ as the set of isomorphism classes of $\mathcal{G}$-torsors. The next corollary settles a task from Section 6.

Corollary 7.21. *Let X be a scheme and $n \geq 0$. Then there exists a natural bijection between $H^1_{\text{ét}}(X, \mathrm{GL}_n)$ and the set of isomorphism classes of rank n vector bundles on X. In particular, $H^1_{\text{ét}}(X, \mathcal{O}^\times) \cong \mathrm{Pic}(X)$.*

Proof. By Lemma 7.19 and the fact that GL_n restricts to the sheaf of $\mathcal{O}_{X_{\text{ét}}}$-automorphisms of $\mathcal{O}^n_{X_{\text{ét}}}$ the H^1 identifies with isomorphism classes of locally free $\mathcal{O}_{X_{\text{ét}}}$-modules on $X_{\text{ét}}$ of rank n. Note that all such $\mathcal{O}_{X_{\text{ét}}}$-modules are quasi-coherent on $X_{\text{ét}}$. By Lemma 7.16 these identify therefore with isomorphism classes on rank n vector bundles on X_{Zar} as claimed. Implicitly, we use the statement from last semester that locally freenees of finite rank can be tested over some faithfully flat cover. □

Torsors provide also many interesting geometric examples.

Lemma 7.22. *Let X be a scheme and G an affine group scheme over X, i.e., a group object G in the category of schemes over X, such that the morphism $G \to X$ is affine. Then each G-torsor $\mathcal{P}$ on Sch/X (more precisely, each h_G-torsor) for the fpqc-topology is representable by some scheme $P \to X$.*

Proof. Locally on X for the fpqc-topology $\mathcal{P}$ is isomorphic to G. By Corollary 7.13 we can conclude that $\mathcal{P}$ is representable. □

Examples for G are $\mathrm{GL}_n, \mathbb{G}_m, \mathbb{G}_a, \mu_n, \underline{H}$ for some finite group H, etc.

Corollary 7.23. *Let X be a scheme and let G an affine, flat group scheme over X. Then the category of G-torsors on Sch/X for the fpqc-topology is equivalent to the category of schemes $P \to X$ with a right G-action $P \times_X G \to P$ such that $P \to X$ is faithfully flat, quasi-compact and the natural map $\Phi\colon P \times_X G \to P \times_X P, (p, g) \mapsto (p, pg)$ is an isomorphism. Moreover, each such P is affine.*

Proof. By Lemma 7.22 each G-torsor $\mathcal{P}$ on Sch/X for the fpqc-topology is representable by some affine morphism $P \to X$. As $G \to X$ is flat and surjective (by the existence of the unit section), we can conclude that $P \to X$ is faithfully flat. The fact that Φ is an isomorphism can be checked locally on X (by Corollary 7.14) and holds for $P = G$. Conversely, note that that Φ is G-equivariant, if G acts on $P \times_X G$ via the right action on G and on $P \times_X P$ via the given right action on the second factor. If Φ is an isomorphism, then the base change of $P \to X$ along itself trivializes the G-torsor $P \times_X P \to P, (p, q) \mapsto p$ over P. As $P \to X$ is assumed to be a cover for the fpqc-topology, this shows that (the sheaf represented by) P is a G-torsor. □

Remark 7.24. If G in Lemma 7.22 is additionally smooth, then by descent each G-torsor $P \to X$ is smooth. This implies that $P \to X$ has sections étale locally on X, i.e., there exists an étale covering $\{X_i \to X\}_{i \in I}$ such that $P(X_i) \neq \emptyset$ for all $i \in I$. We can conclude that the groupoids of fpqc- and étale torsors of G are equivalent. Similarly, fpqc-torsors and fppf-torsors are equivalent if G is an affine, flat, group scheme of finite presentation.

Example 7.25. Let X be a scheme, and consider the fpqc-topology in all examples below. Here is a list of examples for G-torsors for varying groups G

(1) Assume that $G = \mathbb{G}_m$. Then G-torsors are equivalent to the groupoid of line bundles $\mathcal{L}$ on X. Given $\mathcal{L}$ the associated scheme is $\underline{\mathrm{Spec}}_X(\bigoplus\limits_{i\in\mathbb{Z}} \mathcal{L}^i)$. As a special case consider $X = \mathbb{P}^n_{\mathbb{Z}}$ and $\mathcal{L} = \mathcal{O}(-1)$, then $P \cong \mathbb{A}^{n+1}_{\mathbb{Z}} \setminus \{0\}$.
(2) Assume that $G = \underline{H}$ for some finite group H. Then G-torsors identify with finite, étale X-schemes Y equipped with an action of H such that the map $\Phi\colon Y \times H \to Y \times_X Y$ is an isomorphism, or equivalently H acts simply transitive on the geometric fibers of $Y \to X$. As a special case that $X = \mathrm{Spec}(K)$ some field and L/K a finite Galois extension. Then the left action of $H := \mathrm{Gal}(L/K)$ on L defines a right action of $\mathrm{Gal}(L/K)$ on $Y := \mathrm{Spec}(L)$ and $Y \to X$ is an H-torsor.

Let us now fix a scheme X, and affine, flat group scheme G over X and a G-torsor $P \to X$. For simplicity let us assume that $X = \mathrm{Spec}(A)$ is affine, then

$$G = \mathrm{Spec}(\mathcal{O}_G), P = \mathrm{Spec}(B)$$

are affine as well. The group multiplication $m\colon G \times_X G \to G$ on G induces on the A-algebra $\mathcal{O}_G$ a coassociative comultiplication $m^*\colon \mathcal{O}_G \to \mathcal{O}_G \otimes_A \mathcal{O}_G$, and the unit section $e\colon X \to G$ defines a counit $e^*\colon \mathcal{O}_G \to A$ for the comultiplication. In fact, $\mathcal{O}_G$ is a Hopf algebra. The isomorphism $P \times_X G \cong P \times_X P,\ (p,g) \mapsto (pg,p)$ (note the order!) identifies the two projections $p_1, p_2\colon P \times_X P \to P$ with the morphisms

$$\alpha\colon P \times_X G \to P,\ (p,g) \mapsto pg,\ \mathrm{pr}\colon P \times_X G \to P, (p,g) \to p$$

where α denotes the (right) G-action on P. According to Lemma 7.7 we can conclude that descent data for the faithfully flat morphism $P \to X$ identify with pairs of a B-module M with an A-linear map

$$\gamma\colon M \to \mathrm{pr}^*(M) = (B \otimes_A \mathcal{O}_G) \otimes_{B,\mathrm{pr}^*} M \cong M \otimes_A \mathcal{O}_G,$$

which is linear over $\alpha^*\colon B \to B \otimes_A \mathcal{O}_G$, satisfies coassociativity and such that $\mathrm{Id}_M \otimes e^* \circ \gamma = \mathrm{Id}_M$. The datum of an A-module M with such a coassociative, counital morphism $\gamma\colon M \to M \otimes_A \mathcal{O}_G$ is called a comodule for the A-coalgebra $\mathcal{O}_G$. Let us explain how to make the above more explicit if $G = \underline{H}$ for some finite group H. By Example 7.25 $P = \mathrm{Spec}(L) \to X = \mathrm{Spec}(K)$ for some finite Galois extension L/K of fields.

Exercise 7.26. Let us assume the setup from above with $G = \underline{H}$, H a finite group.
(1) Show that $\mathcal{O}_G \cong A \otimes_{\mathbb{Z}} C(H,\mathbb{Z})$, where $C(H,\mathbb{Z})$ denotes the ring of functions $H \to \mathbb{Z}$.
(2) Show that the category of comodules under the A-coalgebra $\mathcal{O}_G$ is equivalent with the category of A-modules M with an A-linear H-action.
(3) Show that the category of $\mathcal{O}_G$-modules M, such that M is B-module and such that as above $\gamma\colon M \to M \otimes_A \mathcal{O}_G$ is α^*-linear identifies with the category of B-modules M with a *semilinear* H-action, i.e., B-modules M with an action of H such that for $m \in M, b \in B, h \in H$ we have
$$h(bm) = h(b)h(m),$$
where $h(b)$ denotes the H-action of $b \in B$.

By Theorem 7.5 we can therefore understand in this case Mod_A concretely by B-modules with a semilinear H-action.

Remark 7.27. Assume that a discrete group H acts on some affine scheme $Y := \mathrm{Spec}(B)$. Then it still makes sense to consider the category of B-modules with semilinear H-action. In general a quotient Y/H does not exist, and even if it exists the map $Y \to Y/H$ need not be an H-torsor (as H may have fixed points on Y). However, the stack quotient $[Y/H]$ exists always and the map $Y \to [Y/H]$ behaves like an H-torsor. Making all notations rigorous one can then describe quasi-coherent sheaves on $[Y/H]$ via B-modules with semilinear H-action. Replacing semilinear actions by comodules one can similarly argue with H replaced by some group scheme.

As an extreme case let us give the following example.

Exercise 7.28. In Remark 7.27 set $Y := \mathrm{Spec}(R)$ and equip it with the trivial action by $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[T,T^{-1}])$. Show that the category $\mathrm{QCoh}([Y/\mathbb{G}_m])$ identifies with the category of $\mathbb{Z}$-graded R-modules. *Hint: This exercise asks in more fancy terms for a description of comodules under the R-coalgebra $\mathcal{O}_{\mathbb{G}_m} := R[T,T^{-1}]$ with coaction $\mathcal{O}_{\mathbb{G}_m} \to \mathcal{O}_{\mathbb{G}_m} \otimes_R \mathcal{O}_{\mathbb{G}_m}$, $T \mapsto T \otimes T$.*

8. Generalities on étale cohomology

We need to prove some generalities on étale cohomology, the first is a commutation with filtered colimits.

Lemma 8.1. *Let $f\colon Y \to X$ be a morphism of schemes, and $\mathcal{F}_i \in \mathrm{Sh}_{\mathrm{Ab}}(Y_{\text{ét}})$, $i \in I$, a filtered system of étale sheaves of abelian groups on Y with colimit $\mathcal{F}$.*

(1) *If Y is qcqs, then for any $n \geq 0$ the natural map*
$$\varinjlim_{i\in I} H^n_{\text{ét}}(Y, \mathcal{F}_i) \to H^n_{\text{ét}}(Y, \mathcal{F})$$
is an isomorphism.

(2) *If f is qcqs, then for any $n \geq 0$ the natural map*
$$\varinjlim_{i\in I} R^n f_*(\mathcal{F}_i) \to R^n f_*(\mathcal{F})$$
is an isomorphism. Here, $f_*\colon \mathrm{Sh}_{\mathrm{Ab}}(Y_{\text{ét}}) \to \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$ denotes the pushforward of étale sheaves.

The proof is quite formal and uses only that each étale covering $\{Y_i \to Y\}_{i\in I}$ of a qcqs scheme Y has a *finite* refinement $\{Z_j \to Y\}_{j\in J}$ such that each $Z_j, Z_j \times_Y Z_i$ are again quasi-compact.

Proof. The proof strategy is similar to Lemma 7.17 and Lemma 4.20. The core case is $n = 0$. Let $\mathcal{U} = \{U_j \to U\}_{j\in J}$ be a covering of some $U \in Y_{\text{ét}}$, and denote by $\varinjlim^p$ the colimit in *presheaves*. If J is finite, then the natural map (notation as in Section 5.1)
$$\varinjlim_{i\in I} \Gamma(\mathcal{U}, \mathcal{F}_i) \to \Gamma(\mathcal{U}, \varinjlim_{i\in I}{}^p \mathcal{F}_i)$$
is bijective as filtered colimits commute with finite limits. Thus,
$$\varinjlim_{i\in I} \mathcal{F}_i(U) \cong (\varinjlim_{i\in I}{}^p \mathcal{F}_i)^+(U)$$
for any $U \in Y_{\text{ét}}$ quasi-compact as then finite coverings are cofinal. If $\mathcal{U} = \{U_j \to U\}_{j\in J}$ is a finite cover with $U_j, U_j \times_U U_k$ quasi-compact, this implies that
$$\varinjlim_{i\in I} \Gamma(\mathcal{U}, \mathcal{F}_i) \to \Gamma(\mathcal{U}, (\varinjlim_{i\in I}{}^p \mathcal{F}_i)^+)$$
is bijective. In particular, we can conclude that
$$\varinjlim_{i\in I} \mathcal{F}_i(U) = ((\varinjlim_{i\in I}{}^p \mathcal{F}_i)^+)^+ = \mathcal{F}(U)$$
whenever U is qcqs. This settles the case that $n = 0$. To get both statements for all n we can now argue as in Lemma 7.17. □

Let now $X_i, i \in I$, be a cofiltered system of schemes such that for each $i \leq j$ the transition morphism $h_{i,j}\colon X_j \to X_i$ is affine. Then $X := \varprojlim_{i\in I} X_i$ exists in schemes. Let $h_i\colon X \to X_i$ be the projection. Let $(\mathcal{F}_i \in \mathrm{Sh}_{\mathrm{Ab}}(X_{i,\text{ét}}), \varphi_{i,j}\colon h^*_{i,j}\mathcal{F}_i \to \mathcal{F}_j)$ be a system of étale sheaves on $(X_i, h_{i,j})$. Set $\mathcal{F} := \varinjlim_{i\in I} h_i^*\mathcal{F}_i$ with transition maps
$$h_i^*\mathcal{F}_i = h_j^* h_{i,j}^* \mathcal{F}_i \xrightarrow{h_j^*(\varphi_{i,j})} h_j^*\mathcal{F}_j$$
for $i \leq j$.

Lemma 8.2. *We use the above notation. Assume moreover that each X_i is qcqs.*

(1) *For each $n \geq 0$ the natural map*
$$\varinjlim_{i\in I} H^n_{\text{ét}}(X_i, \mathcal{F}_i) \to H^n_{\text{ét}}(X, \mathcal{F})$$
is an isomorphism.

(2) *For each $0 \in I$ and $n \geq 0$ the natural map*
$$\varinjlim_{i\in I/0} R^n h_{0,i,*}(\mathcal{F}_i) \to R^n h_{0,*}(\mathcal{F}) \in \mathrm{Sh}_{\mathrm{Ab}}(X_{0,\text{ét}})$$
is an isomorphism. Here, $I/0$ denotes the category of objects over $0 \in I$.

The critical ingredient into the proof is that each qcqs étale map $Y \to X$ is already the base change of some qcqs étale map $Y_i \to X_i$ for some i, which follows because étale maps are locally of finite presentation.

Proof. Note that the first statement for n implies the second for n (by evaluating on qcqs objects $U \in X_{0,\text{ét}}$ and Corollary 5.40). We leave the statement for $n = 0$ as an exercise (*Hint: Use Lemma 8.1 and that each covering of some qcqs object $U \in X_{\text{ét}}$ can be refined by a finite covering, which is pulled back from some X_i.*). Using functorial injective resolutions one can reduce to the case that each $\mathcal{F}_i$ is injective. In this case, one needs to see that $\mathcal{F}$ is acyclic. If $n \geq 1$ and $a \in H^n(X, \mathcal{F})$, then there exists a covering $\{Y_j \to X\}_{j\in J}$ with $a_{|Y_j} = 0$. We may assume that J is finite and each Y_j qcqs. Then this covering is pulled back from some $X_i, i \in I$. We may replace I by I/i and then the usual strategy applies, cf. Lemma 7.17. For a more general presentation, see [Stacks, Tag 03Q4]. □

A distinctive property of the étale site is its invariance under universal homeomorphisms.

Theorem 8.3. *Let $f\colon Y \to X$ be a universal homeomorphism. Then the functor*

$$(Z \to X) \mapsto (f^*(Z) := Z \times_X Y \to Y)$$

induces an equivalence $Y_{\text{ét}} \to X_{\text{ét}}$ of categories, which identifies the étale coverings on both sides. In particular, for $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$ the natural map

$$R\Gamma(X_{\text{ét}}, \mathcal{F}) \to R\Gamma(Y_{\text{ét}}, f^*\mathcal{F})$$

is an equivalence.

Universal homeomorphisms are exactly the integral, radicial(=universally injective) and surjective morphisms of schemes, cf. [Stacks, Tag 04DF].

Proof. Let $Z_1, Z_2 \in X_{\text{ét}}$. We first check that the map

$$\mathrm{Hom}_X(Z_1, Z_2) \to \mathrm{Hom}_Y(f^*(Z_1), f^*(Z_2))$$

is bijective. This claim is local on X, and by glueing morphisms as well on Z_1 and Z_2. Hence, we may assume that X, Z_1, Z_2 are separated, e.g., affine. This implies that Y is separated. Via graphs, we can now identify $\mathrm{Hom}_X(Z_1, Z_2)$ with the open and closed subschemes $\Gamma \subseteq Z_1 \times_X Z_2$ (note that the diagonal of a separated étale map is an open and closed immersion), such that the projection $\Gamma \to Z_1$ is an isomorphism, or equivalently, a universal homeomorphism (as Γ, Z_1 are étale over X and hence $\Gamma \to Z_1$ is automatically étale). The same topological description of morphisms applies over Y and we can conclude. Thus, the functor $X_{\text{ét}} \to Y_{\text{ét}}$ is fully faithful. We show essential surjectivity only in the case that f is additionally a closed immersion (thus defined by some locally nilpotent ideal), refering to [Stacks, Tag 04DZ] for the general case. By the proven fully faithfulness it suffices to construct the lift of some étale morphism $V \to Y$ locally on X. But this follows from the (more general) lemma Lemma 8.4. □

We used the following lemma on lifting smooth and étale morphisms.

Lemma 8.4. *Let S be a scheme and $S_0 \subseteq S$ a closed subscheme. Let $X_0 \to S_0$ be a smooth (resp. étale) morphism and let $x_0 \in X_0$ be a point. Then there exists an open neighborhood U_0 of $x_0 \in X_0$, a smooth (resp. étale) S-scheme U and an isomorphism of S_0-schemes $U_0 \cong U \times_S S_0$.*

Proof. First assume that $X_0 \to S_0$ is smooth. The question is local, and hence we may assume that $S_0 = \mathrm{Spec}(R/J) \subseteq \mathrm{Spec}(R)$ and $X_0 = \mathrm{Spec}(R/J[T_1, \ldots, T_n]/\mathfrak{a}) \subseteq \mathbb{A}^n_{S_0} = \mathrm{Spec}(R/J[T_1, \ldots, T_n])$. By the Jacobian criterion we know that in a neighborhood of x_0 the ideal $\mathfrak{a}$ is generated by some polynomials $f_1, \ldots, f_m \in R/J[T_1, \ldots, T_n]$ such that the Jacobian $(\frac{\partial f_i}{\partial T_j})_{i,j}$ has full rank at x_0. Now, it is clear how to locally lift X_0: Let $\mathfrak{b} \subseteq R[T_1, \ldots, T_n]$ be generated by lifts $g_1, \ldots, g_m$ of the $f_1, \ldots, f_m$. Now, the Jacobian $(\frac{\partial g_i}{\partial T_j})_{i,j}$ will have full rank at $x_0 \in X := \mathrm{Spec}(R[T_1, \ldots, T_n]/\mathfrak{b})$. In particular, X is smooth in a neighborhood U of x. Setting $U_0 := U \times_S S_0$ then solves the claim in the smooth case. Now assume that $X_0 \to S_0$ is étale. By the smooth case we can assume that $X_0 \cong X \times_S S_0$ for some smooth S-scheme X. Now, X_0 is unramified at x_0, which implies that the stalk of $\Omega^1_{X_0/S_0}$ at x vanishes. By Nakayama this implies that $\Omega^1_{X/S}$ vanishes at x_0, which implies that X is unramified (hence étale) in a neighborhood U of x. Then replace X by U and X_0 by $U \times_S S_0$. □

9. The étale cohomology of proper, smooth curves, II

If Y is a scheme, then we denote by $\mathbb{G}_{m,Y}$ the restriction of the representable sheaf $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[T, T^{-1}])$ to the (small) étale site $Y_{\text{ét}}$ of Y.

9.1. $\mathbb{G}_m$**-cohomology of smooth curves.** Let now k be an algebraically closed field, and let $Y \to \mathrm{Spec}(k)$ be a quasi-compact, smooth, connected curve. Taking up Section 6 we want to understand the étale cohomology of X. More specifically, we have to prove that

$$H^i_{\text{ét}}(Y, \mathbb{G}_{m,Y}) = 0, \ i \geq 2.$$

Let $j\colon \eta := \mathrm{Spec}(K) \to X$ be the inclusion of the generic point of X, i.e., $K := k(X)$ is the function field of X.

Lemma 9.2. *The natural maps (specified in the proof) define a short exact sequence*

$$0 \to \mathbb{G}_{m,Y} \to j_*(\mathbb{G}_{m,\eta}) \xrightarrow{\bigoplus \nu_x} \bigoplus_{x \in Y(k)} i_{x,*}\mathbb{Z} \to 0$$

on $Y_{\text{ét}}$.

Proof. Let $U \in Y_{\text{ét}}$ be qcqs. Note that U is again a smooth curve over k, and that $U \times_Y \eta$ is the (finite) set of generic points of U. In particular, the valuations ν_u at point $u \in U(k)$ define an exact sequence

$$0 \to \mathcal{O}^\times(U) \to \mathcal{O}^\times(U \times_Y \eta) \to \bigoplus_{u \in U(k)} \mathbb{Z} \cdot u,$$

where the last term can also be written as $\bigoplus_{x \in Y(k)} \Gamma(x \times_Y U, \underline{\mathbb{Z}}) = \bigoplus_{x \in Y(k)} i_{x,*}(\underline{\mathbb{Z}})(U)$. By Lemma 8.1 we see

$$\bigoplus_{x \in Y(k)} (i_{x,*}(\underline{\mathbb{Z}})(U)) = (\bigoplus_{x \in Y(k)} i_{x,*}(\underline{\mathbb{Z}}))(U).$$

Because valuations are invariant under étale morphism of curves (uniformizers map to uniformizers), we obtain an exact sequence

$$0 \to \mathbb{G}_{mY} \to j_*(\mathbb{G}_{m,\eta}) \xrightarrow{\bigoplus \nu_x} \bigoplus_{x \in Y(k)} i_{x,*}\mathbb{Z}$$

of étale sheaves. As Weil divisors on smooth curves are Cartier divisors, they are locally principal, which implies that the morphism $\bigoplus \nu_x$ is indeed a surjection of étale sheaves. □

In order to calculate $Rj_*(\mathbb{G}_{m,\eta})$ we will need the following consequence of Tsen's theorem.

Theorem 9.3. *We have*

$$H^i_{\text{ét}}(\mathrm{Spec}(K), \mathbb{G}_{m,\eta}) = \begin{cases} K^\times, \ i = 0 \\ 0, \ i > 0 \end{cases}$$

We will prove Theorem 9.3 later, when discussing Galois cohomology in a bit more detail. The cases $i = 0$ and $i = 1$ are easy or follow from Corollary 7.21.

In our situation, we get the following corollary.

Corollary 9.4. *We have*

$$j_*(\mathbb{G}_{m,\eta}) = Rj_*(\mathbb{G}_{m,\eta}).$$

Proof. By Corollary 5.40 it suffices to check that for any qcqs $U \in Y_{\text{ét}}$ we have

$$H^i_{\text{ét}}(\eta \times_Y U, \mathbb{G}_m) = 0$$

for $i > 0$. Note that $\eta \times_Y U = \coprod_{i=1}^{n} \mathrm{Spec}(K_i)$, where the K_i are the function fields for the different connected components of U. Hence, we can apply Theorem 9.3 with K replaced by the K_i and conclude. □

We can now settle Task 6.5 (and thus, modulo Theorem 9.3, the proof of Theorem 6.6 is finished).

Theorem 9.5. *We have*

$$H^i_{\text{ét}}(Y, \mathbb{G}_m) = \begin{cases} \mathcal{O}_Y^\times(Y), \ i = 0 \\ \mathrm{Pic}(Y), \ i = 1 \\ 0, \ i \geq 2, \end{cases}$$

and for each $n \in \mathbb{Z}$ *invertible in* k, *we have a natural exact sequence*

$$0 \to \overbrace{\mu_n(Y)}^{=\mathbb{Z}/n(1)(Y) = \mu_n(k) \cong \mathbb{Z}/n} \to \mathcal{O}_Y^\times(Y) \xrightarrow{n} \mathcal{O}_Y^\times(Y) \to H^1_{\text{ét}}(Y, \mu_n) \to \mathrm{Pic}(Y) \xrightarrow{n} \mathrm{Pic}(Y) \to H^2_{\text{ét}}(Y, \mathbb{Z}/n(1)) \to 0$$

and $H^i_{\text{ét}}(Y, \mathbb{Z}/n(1)) = 0$ *for* $i \geq 3$.

Proof. By Corollary 9.4 and Theorem 9.3 we know

$$R\Gamma_{\text{ét}}(Y, j_*(\mathbb{G}_{m,\eta})) = R\Gamma_{\text{ét}}(Y, Rj_*(\mathbb{G}_{m,\eta})) = R\Gamma_{\text{ét}}(\mathrm{Spec}(K), \mathbb{G}_m) \cong K^\times[0].$$

Using that Y was assumed to be qcqs and Lemma 8.1 we get

$$R\Gamma_{\text{ét}}(Y, \bigoplus_{y\in Y(k)} i_{y,*}(\underline{\mathbb{Z}})) \cong \bigoplus_{y\in Y(k)} R\Gamma_{\text{ét}}(Y, i_{y,*}(\underline{\mathbb{Z}})).$$

Because k is algebraically closed, we can conclude

$$i_{y,*}(\underline{\mathbb{Z}}) \cong Ri_{y,*}(\underline{\mathbb{Z}})$$

from Corollary 5.40 and hence

$$R\Gamma_{\text{ét}}(Y, \bigoplus_{y\in Y(k)} i_{y,*}(\underline{\mathbb{Z}})) = \bigoplus_{y\in Y(k)} \underline{\mathbb{Z}}\cdot y[0]$$

This implies all statements, thanks to the long exact sequence associated with Lemma 9.2. □

Corollary 9.6. *If $n \in \mathbb{Z}$ is invertible in k, then $H^i_{\text{ét}}(\mathbb{A}^1_k, \mathbb{Z}/n(1)) = 0$ for $i > 0$ and $H^0_{\text{ét}}(\mathbb{A}^1_k, \mathbb{Z}/n(1)) \cong \mu_n(k)$.*

Recall from Lemma 6.7 that this statement is falls if n is a prime not invertible in k.

Proof. This follows from Theorem 9.5 because $\mathcal{O}^*(\mathbb{A}^1_k) = k^\times$ and $\mathrm{Pic}(\mathbb{A}^1_k) = 0$. □

At this moment we don't have the machinery available to proof $H^i_{\text{ét}}(\mathbb{A}^m_k, \mathbb{Z}/n) = 0$ if $i > 0$, n is invertible in k and $m \geq 2$.

9.7. **The Brauer group of a field.** In this section we follow [8, Arcata, III.1]. Let K be a field.

Definition 9.8. A central simple K-algebra is a finite dimensional K-algebra (not necessarily commutative), such that the following equivalent conditions are satisfied:

(1) A has center K and no non-trivial two sided non-trivial ideal,
(2) there exists a finite Galois extension L/K such that $A_L := A \otimes_K L$ is isomorphic to some matrix algebra $M_n(L), n \geq 0$ ("the extension L splits A"),
(3) $A \cong M_d(D)$ for some $d \geq 0$ and D/K some finite dimensional K-algebra, which is a division algebra with center K.

If $A = M_d(D), A' = M_{d'}(D')$ are central simple algebras with D, D' central division algebras, then A, A' are called (Brauer) equivalent if D and D' are isomorphic.

Definition 9.9. The Brauer group $\mathrm{Br}(K)$ of K is the abelian group of equivalence classes of central simple algebras A with the group structure given by the tensor product, cf. [Stacks, Tag 03R1].

For example, the inverse of $A \cong M_d(D)$ is $M_d(D^{\mathrm{op}})$, where D^{op} denotes the opposite division algebra. Central simple algebras (of fixed dimension n^2) are equivalently torsors on $\mathrm{Spec}(K)_{\text{ét}}$ for the group scheme $G = \mathrm{PGL}_n$. If X is a scheme and $n \geq 0$, then we denote by

$$\mathcal{M}_n(\mathcal{O}_X) = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{O}_X^n)$$

the sheaf of (non-commutative) rings of $\mathcal{O}_X$-linear endomorphisms of $\mathcal{O}_X^n$.

Definition 9.10. We define PGL_n as the group valued functor on schemes, which sends X to the group of automorphisms of $\mathcal{M}_n(\mathcal{O}_X)$ as an $\mathcal{O}_X$-algebra.

Lemma 9.11. *The functor PGL_n is representable by an affine smooth group scheme over $\mathrm{Spec}(\mathbb{Z})$ of relative dimension $n^2 - 1$. Moreover, the conjugation by matrices defines a short exact sequence*

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$$

of Zariski sheaves on $\mathrm{Sch}/\mathrm{Spec}(\mathbb{Z})$ (and hence the sequence is exact a fortiori as a sequence of étale or fpqc sheaves).

We only need this sequence for schemes, which are étale over a field, but we mention its general version.

Proof. The fact that PGL_n is representable by an affine scheme follows easily from the fact that $\mathcal{M}_n(\mathcal{O}_X)$ is a finite, free $\mathcal{O}_X$-algebra (with basis given by the elementary matrices $E_{i,j}$). Moreover, it is clear that PGL_n is of finite presentation. The existence of the "acting by conjugation morphism"

$$\Phi\colon \mathrm{GL}_n \to \mathrm{PGL}_n$$

is clear. Let $A \in \mathrm{GL}_n(\mathcal{O}_X)$ lie in the kernel of Φ. Then on elementary matrices

$$E_{i,j} = A \cdot E_{i,j} \cdot A^{-1},$$

which implies that A is a diagonal matrix, and thus lies in the image of the central morphism $\mathbb{G}_m \to \mathrm{GL}_n,\ t \mapsto t \cdot \mathrm{Id}$. The map $\mathrm{GL}_n \to \mathrm{PGL}_n$ is formally smooth. Indeed, as $\mathrm{GL}_n \to \mathrm{PGL}_n$ are noetherian and of finite type over $\mathrm{Spec}(\mathbb{Z})$ it suffices to check formal smoothness for square-zero thickenings of artinian rings. This can be handled using the Skolem-Noether theorem for matrix algebras [39]. For details, see [6, Exercise 5.5.5]. The surjectivity of $\mathrm{GL}_n \to \mathrm{PGL}_n$ follows now again by the Skolem-Noether theorem. □

Corollary 9.12. *The étale cohomology set $H^1_{\text{ét}}(\mathrm{Spec}(K), \mathrm{PGL}_n)$ identifies with the set of isomorphism classes of central simple K-algebras of dimension n^2.*

Proof. This follows from Lemma 9.11 and Section 7.18. More precisely, we can send a central simple algebra A over K of dimension n^2 to the PGL_n-torsor (on $\mathrm{Spec}(K)_{\text{ét}}$) sending U to $\mathrm{Isom}_U(\mathcal{M}_n(\mathcal{O}_U), \widetilde{A}_U)$. □

To proceed further, we note the following general lemma.

Lemma 9.13. *Let $\mathfrak{X}$ be any topos and $0 \to A \to G \to H \to 1$ a short exact sequence of group objects in $\mathfrak{X}$ such that $A \to G$ is the inclusion of a central subgroup. Then there exists a natural exact sequence (of pointed sets)*

$$H^1(\mathfrak{X}, G) \to H^1(\mathfrak{X}, H) \xrightarrow{\delta} H^2(\mathfrak{X}, A),$$

generalizing the usual connecting morphism if G, H are abelian.

Proof. We only sketch the construction. Let $A \hookrightarrow I$ be an injection with I an injective abelian group object in $\mathfrak{X}$. Using the centrality of $A \to G$ we can construct a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & G' & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

of extension of group, where again $I \to G'$ is central. Concretely, we can construct G' as the quotient of $I \times G$ by the equivalence relation $(ia, g) = (i, ag)$ for $i \in I, a \in A, g \in G$ (the centrality of $A \to G$ is used to check that $(i,g)(i',g') := (ii', gg')$ is a well-defined group structure). An H-torsor P can now be lifted to a G'-torsor P' (using injectivity of I and applying [Stacks, Tag 0CJZ] to the gerbe of liftings of P to a G'-torsor). Quotening G' by its normal subgroup $A \subseteq I$ yields,

$$G'/A \cong I/A \times H,$$

and this $G'/A \times^{G'} P' \cong P_1 \times P$ for some I/A-torsor P_1. The image of P_1 under the connecting homomorphism

$$H^1(\mathfrak{X}, I/A) \to H^2(\mathfrak{X}, A)$$

defines now $\delta([P])$. □

From Lemma 9.11 we can deduce the existence of a natural connecting map

$$\theta_n \colon H^1_{\text{ét}}(\mathrm{Spec}(K), \mathrm{PGL}_n) \to H^2_{\text{ét}}(\mathrm{Spec}(K), \mathbb{G}_m).$$

Our interest in these maps lies in the fact that it makes the a priori unaccessible group $H^2_{\text{ét}}(\mathrm{Spec}(K), \mathbb{G}_m)$ more concrete via PGL_n-torsors respectively central simple algebras (of dimension n^2).

Lemma 9.14. *Let K be a field.*

(1) *For each $n \geq 0$ the map $\theta_n \colon H^1_{\text{ét}}(\mathrm{Spec}(K), \mathrm{PGL}_n) \to H^2_{\text{ét}}(\mathrm{Spec}(K), \mathbb{G}_m)$ is injective.*
(2) *Let $\alpha \in H^2_{\text{ét}}(\mathrm{Spec}(K), \mathbb{G}_m)$, L/K a Galois extension of degree n. If $\alpha_{|\mathrm{Spec}(L)} = 0$, then α lies in the image of θ_n.*
(3) *The θ_n assemble into an isomorphism $\mathrm{Br}(K) \cong H^2_{\text{ét}}(\mathrm{Spec}(K), \mathbb{G}_m)$.*

[39] If k is a field, then any k-linear automorphism of $M_n(k)$ is given by conjugation with some element in GL_n, cf. [Stacks, Tag 074R].

Proof. For the first and third statement, we refer to the literature, e.g., [8, Arcata, III.1]. Let us prove the second statement. Fix a separable closure $\overline{K}$ of K containing L. Set $G := \mathrm{Gal}(\overline{K}/K)$, $H := \mathrm{Gal}(\overline{K}/L) \subseteq G$, which is an open subgroup. Then the morphism

$$f\colon \mathrm{Spec}(L)_{\text{ét}} \to \mathrm{Spec}(K)_{\text{ét}}$$

of topoi identifies with the morphism

$$\pi\colon H - \mathrm{Sets} \to G - \mathrm{Sets}$$

of topoi (for simplicity we suppress the superscript "cont"), such that π^{-1} restricts a G-action to an H-action. In particular, the right adjoint π_*(=induction of a discrete H-module to a discrete G-module) is a the exact functor $\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], -)$ on discrete H-modules. We can deduce that for any H-module M, we have

$$R\Gamma(G, \pi_*(M)) = R\Gamma(H, M),$$

which is also known as "Shapiro's lemma". More precisely,

$$\mathrm{Spec}(K)_{\text{ét}} \cong G - \mathrm{Sets},\ \mathcal{F} \mapsto \varinjlim_{K'/K \text{ finite Galois}} \mathcal{F}(K')$$

with its G-action induced by functoriality. Choosing a basis of L over K yields the a G-equivariant morphism

$$(\overline{K} \otimes_K L)^\times \to \mathrm{GL}_n(\overline{K}),\ x \mapsto \text{ matrix of multiplication by } x \text{ on } \overline{K} \otimes_K L$$

where G acts via its action on $\overline{K}$. We get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{K}^\times & \longrightarrow & (\overline{K} \otimes_K L)^\times & \longrightarrow & (\overline{K} \otimes_K L)^\times/\overline{K}^\times & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{K}^\times & \longrightarrow & \mathrm{GL}_n(\overline{K}) & \longrightarrow & \mathrm{PGL}_n(\overline{K}) & \longrightarrow & 1
\end{array}$$

of G-equivariant short exact sequences, from which we can deduce a commutative diagram (with exact first row)

$$\begin{array}{ccccc}
H^1(G, (\overline{K} \otimes_K L)^\times/\overline{K}^\times) & \longrightarrow & H^2(G, \overline{K}^\times) & \xrightarrow{h} & H^2(G, (\overline{K} \otimes_K L)^\times) \\
\downarrow & & \downarrow{=} & & \\
H^1(G, \mathrm{PGL}_n) & \xrightarrow[\theta_n]{} & H^2(G, \overline{K}^\times) & &
\end{array}$$

with h identifying with the restriction $H^2_{\text{ét}}(\mathrm{Spec}(K), \mathbb{G}_m) \to H^2_{\text{ét}}(\mathrm{Spec}(L), \mathbb{G}_m) \cong H^2_{\text{ét}}(\mathrm{Spec}(K), f_*(\mathbb{G}_m))$ because $f_*(\mathbb{G}_m)(\overline{K}) \cong (\overline{K} \otimes_K L)^\times$ (or rather for $\overline{K}$ replaced by the colimits of finite Galois extensions K'/K in $\overline{K}$). A small diagram chase shows the claim. Let us note that the surjectivity assertion in the third statement is implied by the next lemma and the second statement. □

Lemma 9.15. *Let G be a profinite group and M a discrete G-module. Then the natural map*

$$\varinjlim_{N \subseteq G \text{ open, normal}} H^i(G/N, M^N) \cong H^i(G, M)$$

is an isomorphism for any $i \geq 0$.

As the higher cohomology of a finite group H is killed by $|H|$ (the "méthode de la trace" in Lemma 9.16 shows this), we see that higher continuous group cohomology of profinite groups is always torsion. In particular, $H^i(G, M) = 0$ whenever $i > 0$ and M is a $\mathbb{Q}$-vector space.

Proof. By Section 5.23 the cohomology of G is the cohomology of the site/topos of continuous G-sets S. If S is a finite (continuous) G-set, let us call S "qcqs". Then each continuous G-set is a union of qcqs ones. Thus, the cohomology of G is equivalently the cohomoloy of the site of *finite* continuous G-sets, e.g., by Exercise 5.34. The argument of Lemma 8.1 then applies and shows that $H^i(G, -)$ commutes with filtered colimits for $i \geq 0$. In fact, we can consider the inverse system $\{G/N\}_{N \subseteq G \text{ open, normal}}$ and then the argument for Lemma 8.2 applies as well because any finite continuous G-set is the inflation of a finite G/N-set for some $N \subseteq G$ open, normal. Writing $M = \bigcup\limits_{N \subseteq} M^N$ (as a continuous G-set) reduces to the case that $M = M^N$ for some $N \subseteq G$. Now, the argument for Lemma 8.2 applies. □

Brauer groups are interesting arithmetic invariants of fields, e.g., $\mathrm{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}$. In our situation we are more interested in fields with vanishing Brauer groups as these automatically have cohomological dimension ≤ 1.

Lemma 9.16. *Let K be a field, $\overline{K}$ a separable closure and $G := \mathrm{Gal}(\overline{K}/K)$. Assume that for any finite subextension K'/K of $\overline{K}$, the Brauer group $\mathrm{Br}(K')$ vanishes. Then*

(1) $H^i(G, \overline{K}^\times) = 0$ *for* $i > 0$,
(2) $H^i(G, M) = 0$ *for* $i \geq 2$ *and any torsion discrete G-module M, i.e., K is of cohomological dimension* ≤ 1.
(3) $H^i(G, M) = 0$ *for* $i \geq 3$ *and any discrete G-module M.*

Proof. We show the second statement first. By Lemma 8.1 and the fact that each torsion discrete G-module is the filtered union of its finite G-submodules, we may assume that M is finite. Filtering M by $M \supseteq \ell M \supseteq \ell^2 M \supseteq \ldots \supseteq \ell^n M = 0$, $n \gg 0$, for all primes ℓ we may even assume that M is an $\mathbb{F}_\ell$-vector space for some prime ℓ. Then the G-action on M factors through a finite quotient. Let $H \subseteq G$ be the preimage of an ℓ-Sylow-subgroup of such a finite quotient. Let $\mathrm{Ind}_H^G M := \{f \colon G \to M \mid f(hg) = hf(g) \text{ for all } g \in G,\ h \in H\}$ the induction, with G-action via right translation on g. The composition

$$M \to \mathrm{Ind}_H^G M \to M,\ m \mapsto (g \mapsto gm),\ f \mapsto \sum_{Hg \in H\backslash G} g^{-1} f(g)$$

is a composition of G-equivariant morphisms and on cohomology $H^i, i \geq 0$ the composition is in fact multiplication by $[G : H]$[40]. As $[G : H]$ is by assumption prime to ℓ this implies that $H^i(G, M)$ is zero if and only if $H^i(H, M)$ is zero. Indeed, we have the factorization

$$H^i(G, M) \to \underbrace{H^i(G, \mathrm{Ind}_H^G M)}_{\cong H^i(H,M)} \to H^i(G, M),$$

which is multiplication by $[G : H]$. After enlarging K we may thus assume that $G = H$, which implies that G acts on M by some finite quotient G/N, which is an ℓ-group. But then G/N has a fixed vector in any finite dimensional $\mathbb{F}_\ell$-representation and using devisságe we may assume that $M = \mathbb{F}_\ell$ is actually trivial.[41] If $\ell = \mathrm{char}(K)$ the Artin-Schreier sequence Lemma 6.7 implies $H^i(G, \mathbb{F}_\ell) = 0$ for $i \geq 2$. If $\ell \in k^\times$, then $\mathbb{F}_\ell \cong \mu_\ell$, at least up to potentially replacing K by the argument above with the prime to ℓ-extension $K(\mu_\ell)$. The Kummer sequence and the assumption imply the second claim. The third claim follows from the second by using that the kernel and cokernel of $M \to M \otimes_{\mathbb{Z}} \mathbb{Q}$ are torsion, and that the higher cohomology of $M \otimes_{\mathbb{Z}} \mathbb{Q}$ vanishes, cf. Lemma 9.15. The first statement follows from the third, the assumption on H^2 and the fact that $H^1_{\text{ét}}(\mathrm{Spec}(K), \mathbb{G}_m) = \mathrm{Pic}(\mathrm{Spec}(K)) = 0$, cf. Lemma 7.19. □

Remark 9.17. The proof of Lemma 9.16 shows that if K is any field of characteristic $p > 0$, then

$$H^i(\mathrm{Gal}(\overline{K}/K), M) = 0$$

for $i \geq 2$ and any p-power torsion module M, i.e., K is of p-cohomological dimension ≤ 1.

9.18. **Tsen's theorem and consequences.** We now attack Theorem 9.3. Tsen's theorem singles out a particular arithmetic property of function fields of curves over some algebraically closed field.

Theorem 9.19. *Let k be an algebraically closed field k and K/k a field extension of transcendence degree 1. Then K is C_1, i.e., for each $n \geq 0$ each non-constant homogeneous polynomial $f(T_1, \ldots, T_n) \in K[T_1, \ldots, T_n]$ of degree $d < n$ has a non-trivial zero.*

We follow [8, Arcata, III.2].

Proof. First assume that $K = k(X)$. Write

$$f(T_1, \ldots, T_n) = \sum_{i_1,\ldots,i_n,\ i_1+\ldots+i_n = d < n} a_{i_1,\ldots,i_n} T_1^{i_1} \cdots T_n^{i_n}$$

with $a_{i_1,\ldots,i_n} \in k(X)$. We may assume that each $a_{i_1,\ldots,i_n} \in k[X]$. Set $\delta := \sup\{\deg(a_{i_1,\ldots,i_n})\}$. We are seeking $N \geq 0$ and some polynomials $g_i(X) = \sum_{j=0}^{N} \lambda_{i,j} X^j$ with $\lambda_{i,j} \in k$ (not all zero), such that

$$0 \stackrel{!}{=} f(g_1(X), \ldots, g_n(X)) = \sum_{m=0}^{d \cdot N + \delta} c_m((\lambda_{i,j})_{i=1,\ldots,n,j=0,\ldots,N}) X^m.$$

[40]If $m \in M^G$ is G-invariant, then the claim is clear. The general case follows by using an injective resolution, noting that Ind_H^G preserves injectives (being pushforward for a morphism of topoi)

[41]This reduction step is also called "méthode de la trace".

If $t \in k^\times$, then by homogenity of f of degree d

$$\begin{array}{rc} & t^d(\sum\limits_{m=0}^{d\cdot N+\delta} c_m((\lambda_{i,j})_{i=1,\ldots,n,j=0,\ldots,N})X^m) \\ = & t^d f(g_1(X),\ldots,g_n(X)) \\ = & f(tg_1(X),\ldots,tg_n(X)) \\ = & \sum\limits_{m=0}^{d\cdot N+\delta} c_m((t\lambda_{i,j})_{i=1,\ldots,n,j=0,\ldots,N})X^m, \end{array}$$

which implies that each c_m is a homogeneous polynomial of degree d. The polynomials $c_0,\ldots,c_{d\cdot N+\delta}$ define a non-empty vanishing locus Z in $\mathbb{A}_k^{n(N+1)}$, as $0 \in Z$. Moreover, each irreducible component of Z has dimension $\geq n(N+1) - d\cdot N - \delta - 1$. As $n > d$, this dimension is positive for $N \gg 0$ as desired.

Now assume that K is general and pick some inclusion $k(X) \to K$. Then $K/k(X)$ is algebraic by assumption and hence a filtered colimit of its finite subextensions. It suffices to treat the case that $K/k(X)$ is actually finite. Let $f(T_1,\ldots,T_n) \in K[T_1,\ldots,T_n]$ be a homogeneous polynomial of degree $d < n$. Let $e_1,\ldots,e_m \in K$ be a basis over $k(X)$ and substitute $T_i = \sum\limits_{j=1}^{m} U_{i,j}e_j$ for some $m\cdot n$ indeterminants $U_{i,j}$. The polynomial

$$N_{K/k(X)}(f(T_1,\ldots,T_n)) \in k[U_{i,j} | i=1,\ldots,n, j=1,\ldots,m]$$

is homogeneous of degree $m\cdot n$ because

$$N_{K/k(X)}(f(tT_1,\ldots,tT_n)) = N_{K/k(X)}(t^d f(T_1,\ldots,T_n)) = t^{dm}N_{K/k(X)}(f(T_1,\ldots,T_n))$$

for $t \in k$. By the proven case $k(X)$ we can conclude that $N_{K/k(X)} \circ f$ and hence f has a non-trivial zero. □

Lemma 9.20. *Let K be a C_1-field. Then* $\mathrm{Br}(K') = 0$ *for any finite extension K'/K. In particular, K is of cohomological dimension ≤ 1, cf. Remark 9.17.*

Proof. The argument in Theorem 9.19 implies that K' is C_1, hence we may assume that $K = K'$. By Lemma 9.14 it suffices to check that $H^1_{\text{ét}}(\mathrm{Spec}(K), \mathrm{PGL}_n) = \{1\}$ for any $n \geq 0$, i.e., that each central division algebra D/K is isomorphic to D. Let $\overline{K}$ be a separable closure of K. Choose an isomorphism $\varphi\colon D\otimes_K \overline{K} \cong M_n(\overline{K})$ and set

$$\mathrm{Nrd}_\varphi := \det \circ \varphi.$$

As each $\overline{K}$-automorphism of $M_n(\overline{K})$ is given by conjugation, Nrd_φ is indepentent of φ and $\mathrm{Gal}(\overline{K}/K)$-equivariant. Thus, it descents to the "reduced norm" $\mathrm{Nrd}\colon D \to K$ over K. Now, Nrd is a homogeneous polynomial of degree n in n^2 variables, and thus if $n > 1$ it admits a non-trivial root $x \in D$. But D is a division algebra and by multiplicativity of Nrd the element x cannot be a unit. This is a contradiction as desired. □

We have now finished the proof of Theorem 9.3 and thus our discussion of the étale cohomology of smooth curves.

10. The proper base change theorem

Our main aim for this course is a proof of the proper base change theorem in étale cohomology (or at least presenting large parts of the proof).

Theorem 10.1 (Proper base change theorem)**.** *Let*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

be a cartesian diagram of schemes with f proper. Then for any <u>*torsion*</u> *complex $K \in D^+(Y_{\text{ét}}, \underline{\mathbb{Z}})$ the natural map*

$$g^* Rf_*(K) \to Rf'_*(g'^{,*}K)$$

is an isomorphism.

Remark 10.2.

1. If $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(Y_{\text{ét}})$ is any abelian sheaf and $n \in \mathbb{Z}$ non-zero, then $\mathcal{F}[n] := \ker(\mathcal{F} \xrightarrow{n} \mathcal{F})$ is the n-torsion of $\mathcal{F}$. The sheaf $\mathcal{F}$ is torsion if $\varinjlim_{n\in\mathbb{Z}\ \textit{non-zero}} \mathcal{F}[n] \to \mathcal{F}$ is an isomorphism. Using Lemma 8.1 this condition is satisfied if and only if for any $U \in Y_{\text{ét}}$ <u>qcqs</u> the abelian group $\mathcal{F}(U)$ is torsion. A torsion complex $K \in D(Y_{\text{ét}}, \underline{\mathbb{Z}})$ is a complex such that each $\mathcal{H}^q(K)$, $q \in \mathbb{Z}$, is a torsion sheaf. We will denote by $D_{\mathrm{tor}}(Y_{\text{ét}})$ the category of torsion complexes on Y.
2. We will discuss an example that the proper base change theorem fails for non-torsion coefficients.
3. Theorem 10.1 reduces formally to the case the K is a torsion sheaf (and not a complex of such), using the usual devisságe.

10.3. **Stalks of étale sheaves.** Similar to the case for topological spaces, we will reduce Theorem 10.1 to an assertion on "stalks". For this we have to pass to "points". In the topos-theoretic sense this means the following.

Definition 10.4. Let $\mathfrak{X}$ be a topos. A point of $\mathfrak{X}$ is a morphism $\xi\colon (\mathrm{Sets}) \to \mathfrak{X}$ of topoi. Given a point ξ and $\mathcal{F} \in \mathfrak{X}$ we call $\mathcal{F}_\xi := \xi^{-1}(\mathcal{F})$ the stalk of $\mathcal{F}$ at ξ.

Example 10.5. Let us list some examples for points of topoi.

1. Let T be a topological space, $t \in T$ and $i_t\colon \{t\} \to T$ be the inclusion. Then
$$i_t\colon (\mathrm{Sets}) \cong \mathrm{Sh}(\{t\}) \to \widetilde{T}$$
is a topos-theoretic point. If T is sober, i.e., each closed irreducible subset has a unique generic point, then each point of $\widetilde{T}$ is isomorphic to i_t for some $t \in T$, cf. Remark 10.6.
2. Let X be a scheme. A geometric point of X is a morphism $\xi\colon \mathrm{Spec}(\Omega) \to X$ with Ω a separably closed field. As $\mathrm{Spec}(\Omega)_{\text{ét}} \cong (\mathrm{Sets})$ each geometric point yields a topos theoretic point of $\widetilde{X_{\text{ét}}}$, which we denote again by ξ if no confusion can arise. The conjunction of Remark 10.6 and Remark 10.7 shows that each topos-theoretic point of $\widetilde{X_{\text{ét}}}$ is of this form. Note that if $f\colon \mathrm{Spec}(\Omega') \to \mathrm{Spec}(\Omega)$ with Ω' separably closed, then $\xi, \xi \circ f$ define equivalent morphisms of topoi. In particular, if $x \in X$ denotes the image of ξ and $\overline{k(x)}$ the separable closure of $k(x)$ in Ω, then $\mathrm{Spec}(\overline{k(x)}), \mathrm{Spec}(\Omega)$ define equivalent points of $\widetilde{X_{\text{ét}}}$. Hence, it is usually no harm to assume that $\Omega = \overline{k(x)}$.
3. There exists topoi without any points, cf. [3, Tome 2, Exposé IV.7.4]. If there exists a set $\xi_i\colon (\mathrm{Sets}) \to \mathfrak{X}, i \in I$, of points such that a morphism $\mathcal{F} \to \mathcal{G}$ in $\mathfrak{X}$ is an isomorphism if and only if $\mathcal{F}_{\xi_i} \to \mathcal{G}_{\xi_i}$ is an isomorphism (of sets) for all $i \in I$, then $\mathfrak{X}$ is said to have "enough points". We will check below that étale topoi have enough points ([3, Tome 2, Exposé VIII, Théoréme 7.9]), and give more explicit formulas for the stalks.

The following two remarks are not necessary for understanding the course, but maybe illuminating for understanding the notion of a point of a topos.

Remark 10.6 ([3, Tome 1, Exposé IV, 4.2.3])**.** Let T, S be sober topological spaces. Then each morphism

$$\varphi\colon \widetilde{T} \to \widetilde{S}$$

of topoi is induced by some continuous map $f\colon T \to S$. More precisely, the category (!) $\mathcal{H}om(\widetilde{T}, \widetilde{S})$ of morphisms of topoi is equivalent to the (category associated with the) poset $\mathrm{Hom}_{\mathrm{cont}}(T, S)$, where $f \leq g$ if and only if $\overline{\{f(t)\}} \subseteq \overline{\{g(t)\}}$ for all $t \in T$.[42]

Remark 10.7 ([Stacks, Tag 0BN5])**.** Let G, H be profinite groups with classifying topoi BG, BH of discrete G-resp. H-sets. Then the category

$$\mathcal{H}om(BG, BH)$$

is equivalent to the category $\mathcal{H}om(G, H)/H$ with objects the continuous group homomorphisms $\varphi\colon G \to H$ and $\mathrm{Hom}(\varphi, \psi) = \{h \in H \mid h\varphi(g)h^{-1} = \psi(g) \text{ for all } g \in G\}$ for two continuous group homomorphisms $\varphi, \psi\colon G \to H$.

Lemma 10.8. *Let X be a scheme.*

1. *If Ω is a separably closed field, and $\xi\colon \mathrm{Spec}(\Omega) \to X$ a morphism of schemes with image $x \in X$, then*
$$\mathcal{F}_\xi = \varinjlim_{\mathrm{Spec}(\Omega) \to U \to X} \mathcal{F}(U)$$
for any $\mathcal{F} \in \widetilde{X_{\text{ét}}}$. Here, the colimit is taken over the "étale neighborhoods of ξ", i.e., the category of diagrams $\mathrm{Spec}(\Omega) \to U \to X$ with $U \in X_{\text{ét}}$ and morphisms given by morphisms in U, which respect the morphism from $\mathrm{Spec}(\Omega)$ and the morphism to X.
2. *The topos $\widetilde{X_{\text{ét}}}$ as enough points, as witnessed by the family of morphisms $\overline{x}\colon \mathrm{Spec}(\overline{k(x)}) \to X$ with $x \in X$ and $\overline{k(x)}$ some separable closure of $k(x)$.*

Proof. The first formula follows by evaluating the general formula for the pullback in Lemma 5.21. Let us check the second statement. Assume that $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of étale sheaves. First assume that $\varphi_{\overline{x}}\colon \mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ is injective for any of the chosen points $\overline{x}$. Assume that $U \in X_{\text{ét}}$ and $s_1, s_2 \in \mathcal{F}(U)$ such that $\varphi_U(s_1) = \varphi_U(s_2)$. It suffices to show that for each $u \in U$ there exists an étale morphism $V_u \to U$ with u in the image such that $\varphi(s_{1|V_u}) = \varphi(s_{2|V_u})$. Let $x \in X$ be the image of u. Then we can lift $\mathrm{Spec}(\overline{k(x)}) \to X$ to a morphism $\mathrm{Spec}(\overline{k(x)}) \to U$ with image u as $\overline{k(x)}$ is separably closed and $U \to X$ étale. As $\varphi_{\overline{x}}\colon \mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ is injective there exists some étale neighborhood $\mathrm{Spec}(\overline{k(x)}) \to V_u \to X$ refining $\mathrm{Spec}(\overline{k(x)}) \to U \to X$ such that $s_{1|V_u} = s_{2|V_u}$. As u lies in the image of $V_u \to U$, this finishes the proof. The argument for surjectivity of $\varphi\colon \mathcal{F} \to \mathcal{G}$ if each $\varphi_{\overline{x}}$ is surjective, is similar. □

Using points we can now easily establish excision for étale cohomology. Let us note that if $f\colon Y \to X$ is étale, then $\widetilde{Y_{\text{ét}}}$ identifies with the slice topos $\widetilde{X_{\text{ét}}}/Y$. By Lemma 5.39 the functor $f^*\colon \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}}) \to \mathrm{Sh}_{\mathrm{Ab}}(Y_{\text{ét}})$ admits an exact left adjoint $f_!\colon \mathrm{Sh}_{\mathrm{Ab}}(Y_{\text{ét}}) \to \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$.

Lemma 10.9. *Let*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

be a cartesian diagram of schemes.

1. *If f is étale and $K \in D(Y_{\text{ét}}, \underline{\mathbb{Z}})$, then the natural map $f'_! \circ g'^{,*}K \to g^* \circ f_!K$, adjoint to the composition $g'^{,*}K \to f'^{,*}g^*f_!K \cong g'^{,*}f^*f_!K$ induced by the counit $K \to f^*f_!K$, is an isomorphism.*
2. *If f is a closed immersion and $K \in D(Y_{\text{ét}}, \underline{\mathbb{Z}})$, then f_* is exact and the natural map $g^*f_*K \to f'_*g'^{,*}K$ is an isomorphism.*
3. *If $f = j$ is an open immersion with complement $i\colon Z \to X$ and $K \in D(X_{\text{ét}}, \underline{\mathbb{Z}})$, then the natural maps form a distinguished triangle*
$$j_!j^*K \to K \to i_*i^*K.$$

Proof. By adjunction it suffices to see that the natural map $f^*g_*\mathcal{F} \to g'_*f'^{,*}\mathcal{F}$ is an isomorphism for any $\mathcal{F} \in \widetilde{X'_{\text{ét}}}$. Let $(U \to Y) \in Y_{\text{ét}}$. Then

$$f^*g_*(\mathcal{F})(U \to Y) = g_*(\mathcal{F})(U \to X) = \mathcal{F}(U \times_X X' \to X')$$

while

$$g'_*f'^{,*}\mathcal{F}(U \to Y) = f'^{,*}\mathcal{F}(U \times_Y Y' \to Y') = \mathcal{F}(U \times_Y Y' \to X').$$

[42] Note that if t is a specialization of some t', then for any $\mathcal{F} \in \widetilde{T}$ there exists a natural map $\mathcal{F}_t \to \mathcal{F}_{t'}$ because each open neighborhood of t contains t'.

Now, $U \times_X X' \cong U \times_Y Y'$ as $Y' \cong Y \times_X X'$. This implies the first statement. By the existence of enough points Lemma 10.8 in $X'_{\text{ét}}$ it suffices to treat the case that $X' = \operatorname{Spec}(\Omega)$ for some separably closed field. Let $\mathcal{F} \in \widetilde{Y_{\text{ét}}}$. Assume that $g = \xi \colon \operatorname{Spec}(\Omega) \to X$ factors over the open $X \setminus f(Y)$. Then the étale neighborhoods $\operatorname{Spec}(\Omega) \to U \to X$ of ξ such that $U \times_X Y = \emptyset$ are cofinal among all and hence

$$(f_*(\mathcal{F})_\xi = \{*\} = f'_*(g'^{,*}\mathcal{F})$$

because $Y' = \emptyset$. Now, assume that ξ factors over Y. Then $Y' \cong \operatorname{Spec}(\Omega)$ and ξ factors uniquely over a morphism $\eta \colon \operatorname{Spec}(\Omega)$. Now the étale neighborhoods $\operatorname{Spec}(\Omega) \to U \times_X Y \to Y$ of η for some étale neighborhood $\operatorname{Spec}(\Omega) \to U \to X$ of ξ are cofinal among all (by Lemma 8.4). This implies that

$$(i_*(\mathcal{F}))_\xi \cong \mathcal{F}_\eta$$

for any $\mathcal{F} \in \widetilde{Y_{\text{ét}}}$. The final statement follows from the first two and Lemma 10.8 by base changing along morphism $\operatorname{Spec}(\Omega) \to X$ with Ω a separably closed field. $\square$

Note that for a general étale morphism $f \colon Y \to X$ the natural map $f_! f^* \mathcal{F} \to \mathcal{F}$ is not injective for some $\mathcal{F} \in \operatorname{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$, as follows by the first statement and a reduction to X the spectrum of a separably closed field. In fact, by 10.9 we can see that

$$f_!(\mathcal{F})_\xi \cong \bigoplus_{\eta \colon \operatorname{Spec}(\Omega) \to Y \text{ lift of } \xi \colon \operatorname{Spec}(\Omega) \to X} \mathcal{F}_\eta$$

for any $\mathcal{F} \in \operatorname{Sh}_{\mathrm{Ab}}(Y_{\text{ét}})$ and geometric point $\xi \colon \operatorname{Spec}(\Omega) \to X$.

Lemma 10.8 motivates the following definition.

Definition 10.10. Let $\xi \colon \operatorname{Spec}(\Omega) \to X$ be a geometric point. Then the inverse limit

$$X^{\mathrm{sh}}_\xi := \varprojlim_{\operatorname{Spec}(\Omega) \to U \to X} U$$

of all étale neighborhoods of ξ is called the strict henselization at ξ.

Some remarks are in order.

Remark 10.11. (1) The category of étale neighborhoods admits finite limits and hence is filtered. Moreover, the étale neighborhoods $\operatorname{Spec}(\Omega) \to U \to X$ with U affine are cofinal among all. In particular, X^{sh}_ξ exists as a scheme and is affine.
(2) Let $f_\xi \colon X^{\mathrm{sh}}_\xi \to X$ be the natural morphism. If $\mathcal{F}$ is a sheaf, then $\mathcal{F}_\xi \cong \Gamma(X^{\mathrm{sh}}_\xi, f^*_\xi \mathcal{F})$ as follows from Lemma 10.8 and Lemma 8.2 (applied to a cofinal diagram of affine étale neighborhoods of ξ).
(3) Write $X^{\mathrm{sh}}_\xi = \operatorname{Spec}(A)$. Then A is necessarily a local ring because each $g \colon \operatorname{Spec}(\Omega) \to U$ factors over $\operatorname{Spec}(\mathcal{O}_{U,g(\{0\})})$ the resulting transition maps are local morphisms of local rings. More precisely, $\operatorname{Spec}(\mathcal{O}_{U,g(\{0\})}$ is the inverse limits of all its affine open neighborhoods, and each of this is part of the inverse limit defining X^{sh}_ξ. The closed point of $\operatorname{Spec}(A)$ maps to the image $x \in X$ of ξ. Lemma 8.4 implies then that the residue field of A identifies with the separable closure of $k(x)$ in Ω.
(4) Strict henselizations are particularly useful for calculating fibers of pushforwards. Assume that $f \colon Y \to X$ is qcqs, $K \in D(Y_{\text{ét}}, \underline{\mathbb{Z}})$ and $\xi \colon \operatorname{Spec}(\Omega) \to X$ is a geometric point. Then

$$(Rf_*(K))_\xi \cong R\Gamma(X^{\mathrm{sh}} \times_X Y, K_{|X^{\mathrm{sh}} \times_X Y})$$

by Lemma 8.2.

In the following section we will analyze the rings A such that $X^{\mathrm{sh}}_\xi \cong \operatorname{Spec}(A)$ for some scheme X more closely.

10.12. **Interlude on henselian pairs.** Let us call a morphism $A \to B$ of arbitrary rings essentially étale if B is a localization of an étale A-algebra.

The motivation for this interlude will be the following theorem, which is important for some finer properties of étale cohomology that we want to prove.

Theorem 10.13. *Let A be a local ring. Then the following two properties are equivalent:*
(1) *Every essentially étale, local morphism $A \to B$ to a local ring B is an isomorphism.*
(2) *A is strictly henselian.*

Using that étale morphisms are of finite presentation it is easy to see that if X is any scheme, $\xi\colon \operatorname{Spec}(\Omega) \to X$ a geometric point and $X^{\mathrm{sh}}_{\xi} = \operatorname{Spec}(A)$, then the ring A satisfies the first property of Theorem 10.13.

Let us start by defining the (very useful) henselian property in great generality. For this let A be any commutative ring and let $I \subseteq A$ be an ideal.

Definition 10.14 ([Stacks, Tag 09XE])**.** The pair (A, I) is called an henselian pair if

(1) I is contained in the Jacobson radical of A,
(2) for any monic polynomial $f \in A[T]$ and factorization $\overline{f} = g_0 h_0$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$, there exists a factorization $f = g \cdot h$ in $A[T]$ with g, h monic and $g_0 = \overline{g}, h_0 = \overline{h}$.

Here, the $\overline{(-)}$ refers to the base change along $A \to A/I$. In other words, a pair (A, I) is henselian if Hensel's lemma holds for (A, I).

Definition 10.15. (1) A local henselian ring (or henselian local ring) is a local ring A such that the pair $(A, \mathfrak{m}_A)$ is henselian.
(2) A local ring is strictly henselian if it is henselian and its residue field is separably closed.

There are many equivalent characterizations of henselian pairs and henselian local rings. The next (big) theorem will occupy us for some time. References for it are [Stacks, Tag 09XI], [Stacks, Tag 04GG], [24].

Theorem 10.16. *Let A be a ring and let $I \subseteq A$ an ideal. Then the following are equivalent:*

(1) *(A, I) is henselian.*
(2) *If $A \to A'$ is étale and $\sigma\colon A' \to A/I$ a map of A-algebras, then there exists a map $A' \to A$ of A-algebras lifting σ.*
(3) *For any finite A-algebra B the map $\operatorname{Idem}(B) \to \operatorname{Idem}(B/IB)$ is a bijection.*
(4) *For any integral A-algebra B the map $\operatorname{Idem}(B) \to \operatorname{Idem}(B/IB)$ is a bijection.*
(5) *I lies in the Jacobson radical of A and every monic polynomial of the form*

$$f(T) = T^n(T-1) + a_n T^n + \ldots + a_0$$

with $a_i \in I$ and $n \geq 1$ has a root $\alpha \in 1 + I$. (If this condition is satisfied the root α is unique. This characterization will however not be important for us.)

If A is local, $I = \mathfrak{m}_A$ its maximal ideal and $k := A/I$, then the above conditions are equivalent to the following:

(6) *For each monic polynomial $f \in A[T]$ and root $a_0 \in k$ of $\overline{f}$ with $\overline{f}'(a_0) \in k^\times$ there exists a root $a \in A$ of f with $\overline{a} = a_0$.*
(7) *Each finite A-algebra B is a product of local rings.*
(8) *For each monic polynomial $f \in A[T]$ the ring $B := A[T]/(f)$ is a product of local rings.*
(9) *If $X \to S := \operatorname{Spec}(A)$ is locally of finite type and quasi-finite at $x \in X$, which maps to the special point s of S, then $X_x := \operatorname{Spec}(\mathcal{O}_{X,x})$ is open in X and $X_x \to S$ is finite. (If $X \to S$ is separated, this implies that $X_x \subseteq X$ is open and closed.)*
(10) *For each polynomial $f \in A[T]$ (not necessarily monic!) and root $a_0 \in k$ of $\overline{f}$ with $\overline{f}'(a_0) \in k^\times$ there exists a root $a \in A$ of f with $\overline{a} = a_0$.*

Here, $\operatorname{Idem}(C)$ for a ring C denotes its set of idempotents. Geometrically, $\operatorname{Idem}(C)$ is thus in bijection with open and closed subsets of $\operatorname{Spec}(C)$ by sending an idempotent $e \in B$ to its vanishing locus $V(e)$.

We will prove the theorem in many steps.

Proof of Theorem 10.16 (3) $\Rightarrow$ (1). Assume that (A, I) satisfies (3). Let us first show that I lies in the Jacobson radical of A. Pick $\mathfrak{m} \subseteq A$ maximal and assume that I is not contained in $\mathfrak{m}$. Set $B := A/(I \cap \mathfrak{m})$. Then $\operatorname{Spec}(B) = V(I) \coprod \operatorname{Spec}(k(\mathfrak{m}))$ as $\mathfrak{m} \notin V(I)$. This implies that $\operatorname{Idem}(B) \to \operatorname{Idem}(B/I)$ is not injective, contradiction.

Now let $\overline{f} = g_0 h_0 \in A/I[T]$ be a factorization of a monic polynomial $f \in A[T]$ into monic polynomials over $A/I[T]$ such that g_0, h_0 generate the unit ideal in $A/I[T]$. Set $B := A[T]/(f)$, which is a finite free A-algebra. Now, $B/I \cong A/I[T]/g_0 \times A/I[T]/h_0$ by the Chinese remainder theorem. By assumption we can therefore write $B \cong B_1 \times B_2$ as a product of two finite, locally free A-algebras B_1, B_2 with $B_1/I \cong A/I[T]/g_0, B_2 \cong A/I[T]/h_0$ by lifting the respective idempotents. Let $\alpha\colon B \to B$ be the A-linear map of multiplication by the residue class of T in B. Then $f(T)$ is the characteristic polynomial of α. Note that α preserves the decomposition of $B = B_1 \times B_2$ (as α is B-linear and thus commutes with multiplying by the respective idempotents). Let g, h be the characteristic polynomials of α on the finite locally B-modules B_1 and B_2. Then $f = gh$ and $\overline{g} = g_0, \overline{h} = h_0$ as $B = B_1 \times B_2$ reduces to $B/I = A/I[T]/g_0 \times A/I[T]/h_0$. □

Proof of Theorem 10.16 (3) $\Leftrightarrow$ (4). This is clear as each integral A-algebra is a filtered colimit of finite A-algebras and the functor $\mathrm{Idem}(-)$ commutes with filtered colimits of rings. □

We can use the proven implications to provide a lot of examples for henselian pairs.

Corollary 10.17. *Let (A, I) be a pair of a ring A and an ideal $I \subseteq A$.*

1. *If I is locally nilpotent, then (A, I) henselian.*
2. *If A is I-adically complete, then (A, I) is henselian.*
3. *If (A, I) is henselian and $J \subseteq A$ an ideal with $V(I) = V(J)$ (as sets), then (A, J) is henselian.*
4. *If $A \to C$ is integral, e.g., surjective, and (A, I) henselian, then $(C, I \cdot C)$ is henselian.*

Proof. If I is locally nilpotent, then $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ is a universal homeomorphism. This implies that for each A-algebra B the map $\mathrm{Spec}(B/I) \to \mathrm{Spec}(B)$ is a homeomorphism. By the proven implication for Theorem 10.16, namely (3) $\Rightarrow$ (1), this implies that (A, I) is henselian. If A is I-adically complete, then $A \cong \varprojlim\limits_n A/I^n$ and I lies in the Jacobson radical of A. As I^{n-1}/I^n is locally nilpotent in A/I^n, the pair $(A/I^n, I^{n-1}/I^n)$ is henselian by (1). Applying the definition of being an henselian pair, we see that we can iteratively lift a factorization of a monic polynomial $f \in A[T]$ over A/I into pairwise prime factors (note that the condition that the factors generate the unit ideal can be checked module I/I^n, where it holds). This completes (2). Assume (3). If B is any A-algebra, then we conclude that $V(I \cdot B) = V(J \cdot B)$ as sets as $V(I) = V(J)$ forces that I and J have the same radical. By the proven implication (3) $\Rightarrow$ (1) for Theorem 10.16 we can conclude. Assume (4). If B is an integral C-algebra, then B is an integral A-algebra. This implies the statement by (3) $\Rightarrow$ (1) in Theorem 10.16 we win. □

The following statement only uses the Definition 10.14.

Exercise 10.18. If $(A_i, I_i)_{i \in J}$ is a filtered colimit of henselian pairs, then $(\varinjlim\limits_{i \in J} A_i, \varinjlim\limits_{i \in J} I_i)$ is henselian.[43]

A sample application of the henselian property is the following.

Exercise 10.19. Let p be a prime and let $\mathbb{Z}_p$ be the (p)-adic completion of $\mathbb{Z}$. Then $\mathbb{Z}_p$ contains all $(p-1)$-roots of unity. *Hint: The polynomial $T^p - T$ factors over $\mathbb{F}_p$ into distinct linear factors.*

On the negative side let us note the following.

Remark 10.20. We note that if (A, I) is henselian and $S \subseteq A$ is multiplicative, then in general $(A[S^{-1}], I[S^{-1}])$ is not henselian, e.g., $I[S^{-1}]$ need not lie in the Jacobson radical of $A[S^{-1}]$, already if A is a complete discrete valuation ring.

From now on let us assume for the proof of Theorem 10.16 that A is local and $I = \mathfrak{m}_A$ its maximal ideal. Set $k := A/I$.

Proof of Theorem 10.16 (3) $\Leftrightarrow$ (7). Assume (3). Then B/I is a finite k-algebra, and hence a product of local rings. Lifting the idempotents implies that B is a product $B_1 \times \ldots \times B_n$ of finite A-algebras, such that each $B_i \otimes_A k$ is a local ring. As $\mathrm{Spec}(B_i) \to \mathrm{Spec}(A)$ is closed and A local, this implies that each B_i is a local ring.

Assume (7). Note that the functor $\mathrm{Idem}(-)$ on rings commutes with products as an idempotent in a product is exactly a collection of idempotents for the factors. Hence to check (3) we only have to consider the case that B is a non-zero local finite A-algebra. But then $B \otimes_A k$ is local, non-zero and the claim follows because non-zero local rings have exactly the idempotents 0 and 1. □

Proof of Theorem 10.16 (1) $\Leftrightarrow$ (8). When proving (3) $\Rightarrow$ (1) we only needed the case of $B = A[T]/(f)$ for $f \in A[T]$ monic. This implies that we've already checked (8) $\Rightarrow$ (1). If (1) holds, then we can factor $f(T) = f_1(T) \ldots f_r(T)$ into monic polynomials such that for each $i = 1, \ldots, r$ the ring $A[T]/(f_i(T))$ is local, i.e., $\overline{f_i}(T) \in k[T]$ is a power of an irreducible polynomial, and the f_i are pairwise prime. Indeed, we just have to lift iteratively the similar factorization of $\overline{f}$. Now the natural map $A[T]/(f) \to \prod\limits_{i=1}^{r} A[T]/f_i(T)$ is an isomorphism (by the Chinese remainder theorem or using Nakayama and checking this mod I). □

[43] This implies, for example, that the ring of integers in any algebraic extension of $\mathbb{Q}_p$ is henselian along (p) (but not necessarily p-adically complete).

Proof of Theorem 10.16 (8) $\Leftrightarrow$ (3). Assume (8). Let B be a finite A-algebra. Then $I \cdot B$ lies in the Jacobson radical of B as $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is closed and A local. This implies that $\mathrm{Idem}(B) \to \mathrm{Idem}(B/I \cdot B)$ is injective. Indeed, checking that two idempotents agree means to check that they agree in the residue field of each maximal ideal.

We know that $B/I \cdot B$ is a product of local rings. Let $e \in B/I \cdot B$ be an idempotent, such that its vanishing locus $V(e)$ has one point (i.e. e is 0 in exactly one factor). Let $x \in B$ be a lift of e and let $f \in A[T]$ be a monic polynomial with zero x. Define $C := A[T]/(f(T))$, which comes with the map $C \to B$, $T \mapsto x$. Let $c \in C/I \cdot C$ be the residue class of T. The pullback of $V(c)$ under the map $\mathrm{Spec}(B/I) \to \mathrm{Spec}(C/I)$ is $V(e)$. Now $V(c)$ is an open and closed subset of $\mathrm{Spec}(C/I)$ as C/I is a finite k-algebra and hence (on topological spaces) $V(c) = V(d)$ for some idempotent $d \in C/I$. As by assumption C is a product of local fields we may lift d to an idempotent in C and the image of this idempotent in B will lift e. □

Being more careful with the last arguments shows that actually (5) implies (3) in Theorem 10.16, cf. [Stacks, Tag 0EM0]. Note that (1) $\Rightarrow$ (5) holds trivially.

At this stage we have proven (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) (and mentioned that this is equivalent to (5), when using some more care with the above arguments).

For the next implications we need some preparation.

Remark 10.21. Assume that B is a finite free A-algebra, e.g., $B = A[T]/(f)$ for a monic polynomial $f(T) \in A[T]$. Choose a basis $e_1, \ldots, e_n$ of B. Let $b = \sum\limits_{i=1}^{n} \lambda_i e_i \in B$. Then we can write

$$b^2 - b = \sum_{i=1}^{n} g_i(\lambda_1, \ldots, \lambda_n) e_i$$

for certain polynomials $g_i(T_1, \ldots, T_n) \in A[T_1, \ldots, T_n]$. This implies that the functor $\mathrm{Idem}_{B/A}$ on A-algebras

$$C \mapsto \mathrm{Idem}_{B/A}(C) := \{x \in C \otimes_A B \mid x^2 = x\}$$

of idempotents for (base changes of) B is representable by a scheme over A, written $\mathrm{Idem}_{B/A} \to \mathrm{Spec}(A)$. More concretely, it is represented by

$$\mathrm{Spec}(A[T_1, \ldots, T_n]/(g_1(T_1, \ldots, T_n), \ldots, g_n(T_1, \ldots, T_n))),$$

and hence in particular affine and of finite presentation. If $J \subseteq C$ is a nilpotent ideal, then $\mathrm{Idem}_{B/A}(C) \to \mathrm{Idem}_{B/A}(C/J)$ is bijective as idempotents can uniquely be lifted along nilpotent thickenings (more geometrically $\mathrm{Spec}(C/J \otimes_A B) \to \mathrm{Spec}(C \otimes_A B)$ is a homeomorphism). This implies that $\mathrm{Idem}_{B/A} \to \mathrm{Spec}(A)$ is formally étale and thus represented by an étale A-algebra.

Proof of Theorem 10.16 (2) $\Rightarrow$ (8). Set $B := A[T]/(f)$ for some monic polynomial $f \in A[T]$. It suffices to check that $\mathrm{Idem}(B) \to \mathrm{Idem}(B/I)$ is surjective. Take $e \in \mathrm{Idem}(B/I)$. Set $A' := \mathrm{Idem}_{B/A}$ as the functor of idempotents for B, which was constructed in Remark 10.21. Then $A \to A'$ is étale. By definition, $e \in \mathrm{Idem}(B/I)$ defines an A/I-valued point of A', i.e., a morphism $\sigma \colon A' \to A/I$ of A-algebras. By assumption σ can be lifted to a morphism $A' \to A$. But this exactly means that $e \in \mathrm{Idem}(B/I)$ can be lifted to an idempotent in B. □

Let us now check the implications that are left and easy before moving to the hard implications (6) $\Rightarrow$ (2) and (7) $\Rightarrow$ (9).

Proof of Theorem 10.16 (9) $\Rightarrow$ (2), (2) $\Rightarrow$ (10), (10) $\Rightarrow$ (6). The implication (10) $\Rightarrow$ (6) is trivial. Thus assume (9). Let $X := \mathrm{Spec}(A') \to S = \mathrm{Spec}(A)$ be an étale morphism, and let $x \in X(k)$ be a point defined by a morphism $A' \to A/I = k$ of A-algebras. Now, (9) implies that X_x is open in X and that $X_x \to S$ is finite. Replacing X by X_x we reduce to that case that $A \to A'$ is finite étale and A' local with residue field k. But then the map $k \to k \otimes_A A'$ is necessarily an isomorphism, and hence $A \to A'$ is an isomorphism (by Nakayama it is a surjection of A onto a non-zero finite free A-module, hence an isomorphism). This proves (2). Now we assume (2) and prove (10). If $f \in A[T]$ is a polynomial as in (10), then $C := A[T, 1/f']/(f)$ is étale at the k-rational point $a_0 \in \mathrm{Spec}(C/I)$. Thus a suitable open neighborhood $\mathrm{Spec}(A')$ of a_0 is étale over $\mathrm{Spec}(A)$. By (2) we may find a section $A' \to A$, and then the image of $T \in C$ under $C \to A' \to A$ yields the desired solution of f. □

We are now left with the two implications (6) $\Rightarrow$ (2) and (7) $\Rightarrow$ (9) (we will follow [24, Chapitre VII, Proposition 3] now). To prove them we need Zariski's main theorem.

Theorem 10.22 (Zariski's main theorem)**.**
(1) *Let $f: X \to S$ be a morphism of schemes. Let $U \subseteq X$ be the set of points where f is quasi-finite, i.e., those $x \in X$ such that x is open in $f^{-1}(x)$. Then U is open in X.*
(2) *Let $f: X \to S$ be a quasi-finite and separated morphism. Then f is quasi-affine. More precisely, if S is additionally qcqs, then there exists a factorization $X \xrightarrow{j} S' \xrightarrow{\pi} S$ with j a quasi-compact open immersion and π a finite morphism.*

Proof. This is [Stacks, Tag 01TI] and [Stacks, Tag 05K0]. The affine version of the theorem states the more precise version that if $A \to B$ is a morphism of finite type, $A' \subseteq B$ the integral closure of A in B and $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ quasi-finite at the point $x \in \mathrm{Spec}(B)$, then the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A')$ is an open immersion near x. This version is proven in [24, Chapitre IV.Théorème 1]. □

Unfortunately, the proof of Theorem 10.22 is rather long (but not exceptionally difficult, i.e., it does not use methods that we did not use yet). For this reason we have to skip its proof, and only mention the following analog for complex analytic spaces.

Exercise 10.23. Let $f: Y \to X$ be a morphism of complex analytic spaces. Show that the set of points $y \in Y$ such that f is quasi-finite at y is open. *Hint: Embed locally $Y \to X \times \mathbb{C}^n$, and use projections as well as induction on n. If $n = 1$ use Weierstraß preparation.*

We can now prove one of the remaining implications.

Proof of Theorem 10.16 (7) ⇒ (9)*.* We assume (7), i.e, that each finite A-algebra B is a product of local rings. Now let $f: X \to S = \mathrm{Spec}(A)$ be a morphism of locally finite type, which is quasi-finite at $x \in f^{-1}(s)$, where $s = \mathrm{Spec}(k)$ is the closed point of S. We may assume that X is affine, and then by the first point of Theorem 10.22 that f is quasi-finite as the quasi-finite locus is open. By Theorem 10.22 we can then assume that X is an open subset of some S-scheme S' with $S' = \mathrm{Spec}(B) \to S$ finite. By assumption S' is a disjoint union of spectra of local rings. Now, $X_x \subseteq S'_x$, but $X \cap S'$ is open, contains x and $x \in f^{-1}(s)$. This forces $X_x = S'_x$. In particular, X_x is open and closed in X as S'_x is open and closed in S. □

Before proving (6) ⇒ (2) we give the following definition.

Definition 10.24. Let A' be an A-algebra. Then A' is called a standard étale A-algebra if $A' \cong A[T, 1/g]/(f)$ for some monic polynomial $f \in A[T]$ and some $g \in A[T]$, such that the derivative f' of f is invertible in A'.

Clearly, each standard étale A-algebra is étale by the Jacobian criterion. Indeed, as the derivative f' of f is invertible in A' we exactly make the Jacobian criterion work and A' is smooth over A. But A' is also quasi-finite over A, and hence étale. We now prove an easy case of (6) ⇒ (2).

Proof of Theorem 10.16, (6) ⇒ (2) *for standard étale A-algebras.* Assume that $A' = A[T, 1/g]/(f(T))$ is standard étale and that $\sigma: A' \to k$ is an A-algebra morphism. This implies that $\overline{f} \in k[T]$ has a root a_0, and moreover that $\overline{f}'(a_0) \neq 0$ in k. By our assumption (6) we can lift a_0 to a zero $a \in A$. Sending T to $a \in A$ defines a morphism $A[T]/(f) \to A$ and as $\overline{f}'(a_0) \in k^\times$ and A is local, this morphism will factor over A'. Hence, we produced the desired lift. □

Thus we have reduced (6) ⇒ (2) (and thus the final step of the proof of Theorem 10.16) to the following general theorem, cf. [24, Chapitre 5, Théorème 1], [Stacks, Tag 00UE].

Theorem 10.25. *Let $g: X \to S$ be a morphism and $x \in X$ a point such that g is étale at x. Then there exists an open neighborhood $U = \mathrm{Spec}(A')$ of x with image contained in an open affine $\mathrm{Spec}(A) \subseteq S$ such that A' is a standard étale A-algebra.*

Proof. To explain the statement let us first assume that $S = \mathrm{Spec}(k)$ is the spectrum of a field k. Then X is a disjoint union of spectra of finite separable field extensions l/k. Localizing on X we assume that $X = \mathrm{Spec}(l)$ for l/k a finite separable field extension. Now the theorem of the primitive element implies that l is generated by an element $x \in l$ as a k-algebra. Thus $l \cong k[T]/f(T)$ for the monic minimal polynomial f of x. Now f is separable which implies that $f'(x) \neq 0 \in l$, and thus $l \cong k[T, 1/f']/f$ is standard étale.

Now assume that S is a general scheme. Let $s \in S$. By spreading out an isomorphism to a standard étale of $X \times_S S_s$ from S_s to an open neighborhood (this is possible by first shrinking X, S to be qcqs and g to be locally of finite presentation), we may assume that $S = \mathrm{Spec}(A)$ for some some local ring A with maximal ideal I. Let $k := A/I$ be the residue field of A, corresponding to the closed point $s \in S$. We want now to lift from the case $S = \mathrm{Spec}(k)$ handled before. As X is étale at x it is quasi-finite at x. We may replace X by an open neighborhood of x and hence by

Theorem 10.22 we may assume that $X \to S$ is quasi-finite, and in fact an open subset of a finite S-scheme S'. Replacing X by S' reduces to the case that $g\colon X = \mathrm{Spec}(B) \to S = \mathrm{Spec}(A)$ is finite. Now, $B/I \cdot B$ is a finite product of local k-algebra, one of which is $k(x)$ as g is étale at x. The extension $k(x)/k$ is finite separable, and hence $k(x) = k[T]/f(T)$ for some monic separable polynomial $f \in k[T]$. Let $b \in B/I$ be the element, which is zero in all factors, except for $k(x)$, and such b maps to the residue class of T for the factor $k(x)$. Let $c \in B$ be a lift of b and set $C := A[c] \subseteq B$. Let $y \in Y := \mathrm{Spec}(C)$ be the image of $x \in X := \mathrm{Spec}(B)$ under the map $h\colon X \to Y$. We claim that the map $X_x \to Y_y$ is an isomorphism. First, x is the only preimage of y. Indeed, if $z \in \mathrm{Spec}(B/I \cdot B)$ maps to x, then the element $c \in C$ is invertible in $k(z)$, and hence by definition of c we can conclude that $z = x$. The morphism $h\colon X \to Y$ is finite (as $X \to S$ is finite), and as shown, $h^{-1}(y) = \{x\}$. This implies that $X_x \cong Y_y \times_Y X$ and hence that $X_x \to Y_y$ is finite. Note that the map $k(y) \to k(x)$ is surjective (because c mapsto to T), and hence an isomorphism. By Nakayama this implies that $X_x \to Y_y$ is a closed immersion. However, $C \to B$ is injective and hence $X \to Y$ is schematically dominant. Because $X_x \cong Y_y \times_Y X$ this implies that $X_x \to Y_y$ is an isomorphism as claimed. As B is finite over A this implies that $C[1/d] \cong B[1/d]$ for some element $d \in C$ with $d(y) \neq 0$ (in $k(y)$). This implies that we may replace B by C and assume that $B = A[b]$ is generated by some element $b \in B$ (and finite over A). Let $r := \dim_k B/I \cdot B$. Then $1, \bar{b}, \ldots, \bar{b}^{r-1}$ generate $B/I \cdot B$, which implies by the Nakayama lemma that $1, b, \ldots, b^{r-1}$ generate B as an A-module. This implies that b^r can be expressed in terms of the $1, b, \ldots, b^{r-1}$ and hence that we have a surjection $A[T]/(f(T)) \twoheadrightarrow B$ for some monic polynomial $f \in A[T]$. As $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is étale at x, the derivative f' cannot vanish at x, and hence we obtain a surjection $A[T, 1/f']/(f(T)) \to B$. The locus where $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is étale, is open in $\mathrm{Spec}(B)$ (by the Jacobian criterion), and hence there exists some $g \in A[T]$ such that $B[1/g]$ is étale and $x \in D(g) \subseteq \mathrm{Spec}(B)$. We obtain the surjection $A[T, 1/(f'g)]/(f) \twoheadrightarrow B[1/g]$ whose source is standard étale. Geometrically, $\mathrm{Spec}(B[1/g]) \to \mathrm{Spec}(A[T, 1/(f'g)]/(f))$ is a closed immersion and étale by Lemma 10.26. But this implies that $h\colon \mathrm{Spec}(B[1/g]) \to \mathrm{Spec}(A[T, 1/(f'g)]/f(T))$ is the inclusion of an open and closed subset. We may now localize $A[T, 1/(f'g)]/f(T)$ at the respective idempotent e to reduce to the case that h is an isomorphism. Note that this localization does not destroy the property of being standard étale as we can equivalently also localize at $(f'g)^n e \in A[T]$ for some $n \geq 0$. □

In the above proof we used the following useful observation.

Lemma 10.26. *Let $Y \xrightarrow{g} X \xrightarrow{f} S$ be morphisms of schemes and assume that f is unramified. Then g is unramified (resp. étale resp. smooth) if and only if $f \circ g$ is so.*

Proof. The argument is standard by factoring $g\colon Y \to X$ over its graph $Y \to X \times_S Y$ and using that the graph is a base change of the diagonal of f. The essential point is that the diagonal Δ_f of f is an open immersion as f is unramified, and hence étale. □

We can now prove our initial aim Theorem 10.13.

Proof of Theorem 10.13. Set $k := k(\mathfrak{m}_A)$. Assume that A satisfies (1). We check Theorem 10.16 (2), which then implies that A is henselian. Because as $A/I = k$ is a field we may localize the given étale A-algebra A' at $\ker(\sigma)$ and apply the assumption. Assume now that k'/k is a finite, separable field extension. By Lemma 8.4 (applied to $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(k)$) there exists an étale A-algebra A' with $A' \otimes_A k \cong k'$ as k-algebras. By localizing we may assume that A' is local and essentially étale over A. By assumption $A \to A'$ admits then a section, but after base change to k this implies that $k \to k'$ has a section, i.e., $k = k'$. Thus, k is separably closed.

Conversely, assume that A is strictly henselian and $A \to A'$ is a local, essentially étale morphism. Write $A' = B_{\mathfrak{q}}$ for an étale A-algebra B and some $q \in \mathrm{Spec}(B)$ (lying necessarily over $\mathfrak{m}_A$). By Corollary 10.17 (1) $\Rightarrow$ (9) we get that $A \to A'$ is actually finite, and hence finite free. As $A \to A'$ is étale and A' local we can conclude that $k' := k(\mathfrak{m}_{A'}) \cong k \otimes_A A'$. As k is separably closed, we must have $k = k'$. But this implies that $A \to A'$ is surjective by Nakayama's lemma. Being a flat local morphism of local rings it must finally be injective, and hence bijective. This finishes the proof. □

Let us mention a non-obvious cohomological consequence of Theorem 10.16.

Lemma 10.27. *Let $f\colon Y \to X$ be an integral morphism of schemes. Then the pushforward $f_*\colon \mathrm{Sh}_{\mathrm{Ab}}(Y_{\text{ét}}) \to \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$ is exact, and commutes with any base change $X' \to X$.*

In particular, the statement of Theorem 10.1 holds if f is an integral morphism (and then without the restriction to torsion complexes).

Proof. We may assume that X (and consequently Y) is affine. Let $\xi\colon \mathrm{Spec}(\Omega)\to X$ be a geometric point and $\mathcal{F}\in \mathrm{Sh}_{\mathrm{Ab}}(Y_{\text{ét}})$. Then

$$(Rf_\ast(\mathcal{F}))_\xi \cong R\Gamma(X^{\mathrm{sh}}_\xi \times_X Y, \mathcal{F}_{|X^{\mathrm{sh}}_\xi\times_X Y})$$

by Remark 10.11. Using Lemma 8.2 again (and Lemma 5.18 to write $\mathcal{F}$ as a colimit of sheaves, which arise via pullback from finite level), we may assume that $Y\to X$ is finite. By Theorem 10.16 we see that $X^{\mathrm{sh}}_\xi \times_X Y$ is a product of strictly henselian local rings. But Theorem 10.13 implies that $\Gamma(\mathrm{Spec}(A),-)$ is exact for any strictly henselian local ring A. More precisely, we see that

$$f_\ast(\mathcal{F})_\xi \cong \prod_{i=1}^n \mathcal{F}_{\eta_i},$$

where $\eta_1,\ldots,\eta_n\colon \mathrm{Spec}(\Omega)\to Y$ are the pairwise different lifts of $\xi\colon\mathrm{Spec}(\Omega)\to X$ to Y. This formula shows that $f_\ast$ commutes with any base change $X'\to X$ because it commutes with base change to geometric points. □

After having proven (most of) Theorem 10.16 we check for completeness that each analytic algebra (in the sense of Definition 3.30) is henselian.

Lemma 10.28. *Let A be an analytic algebra. Then A is (strictly) henselian.*

For alternative (more direct) proofs see [24, Chapitre VII, Proposition 4] or [10, Lemma 1.7].

Proof. We may assume that $A=\mathbb{C}\{z_1,\ldots,z_n\}$ (by Corollary 10.17 (4)). Let A' be an essentially étale A-algebra. Then we can find a quasi-finite analytic A-algebra B with an injection $A'\to B$ of A-algebras (embed $\mathrm{Spec}(A')$ into some $\mathbb{A}^m_{\mathrm{Spec}(A)}$, then we can use the exact same equations to define a complex analytic space Z in $\mathbb{C}^{n+m}$ and take B as the local ring in Z defined by the closed point of $\mathrm{Spec}(A')$). By Proposition 3.45 we can conclude that A' is a finite A-algebra (using that A is noetherian) and then that $A\to A'$ is an isomorphism as A' has residue field $\mathbb{C}$. By Theorem 10.13 this implies that A is (strictly) henselian. □

Henselian pairs exist in abundance.

Lemma 10.29 ([Stacks, Tag 0A02])**.** *Let A be a ring and $I\subseteq A$. Then there exists a unique (up to unique isomorphism) A-algebra A^h_I such that A^h_I is henselian along $I\cdot A^h_I$ and the natural map $A/I\to A^h_I/I\cdot A^h_I$ is an isomorphism. Moreover, it B is any A-algebra which is henselian along $I\cdot B$, then there exists a unique morphism $A^h_I\to B$ of A-algebras.*

The A-algebra $A^h_I/I\cdot A^h_I$ is called the henselization of A in I.

Proof. We can define A^h_I is the colimit over all factorizations $A\to B\to A/I$ with $A\to B$ étale such that $B/I\cdot B\cong A/I$. By Theorem 10.16 we can conclude that A^h_I is henselian along $I\cdot A^h_I$ and that it satisfies the universal property. □

For a generalization of the procedure of henselization, which also includes strict henselizations, see [4, Definition 2.2.10]. Namely, for any ind-étale A/I algebra C, e.g., $A/I=k$ a field and $C=\overline{k}$ a separable closure, one can consider the colimit $\mathrm{Hens}_A(C)$ of $B's$ running over diagrams $A\to B\to C$ of A-algebras with $A\to B$ étale.

10.30. **Étale cohomology with $\mathbb{Z}$-coefficients.** In this section we want to calculate some examples of étale cohomology with $\mathbb{Z}$-coefficients and in particular explain why the proper base change theorem fails for non-torsion coefficients.

We start with the following observation on (continuous) group cohomology.

Lemma 10.31. *Let G be a profinite group. Then*

$$H^i(G,\mathbb{Z})\cong\begin{cases}\mathbb{Z}, & i=0\\ 0, & i=1\\ H^{i-1}(G,\mathbb{Q}/\mathbb{Z}), & i\geq 2.\end{cases}$$

Here, the action of G on $\mathbb{Z},\mathbb{Q}/\mathbb{Z}$ is trivial.

Proof. Consider the short exact sequence

$$0\to\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}\to 0$$

of discrete G-modules (with trivial action). By 9.15 we can conclude that $H^i(G,\mathbb{Q})=0$ for $i\geq 1$. Using that $\mathbb{Q}=\mathbb{Q}^G\to\mathbb{Q}/\mathbb{Z}=\mathbb{Q}/\mathbb{Z}^G$ is surjective, the claim follows. □

Remark 10.32. By 7.19 $H^1(G,\mathbb{Z}))$ classifies $\mathbb{Z}$-torsors P in the topos of discrete G-sets. As $P \cong \mathbb{Z}$ (as sets with simply transitive $\mathbb{Z}$-action) one checks easily that $H^1(G,\mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{cont}}(G,\mathbb{Z})$, and the latter group vanishes because G is profinite and $\mathbb{Z}$ discrete.

We can draw the following consequence for étale cohomology.

Lemma 10.33. *Let X be a scheme and $i\colon \{x\} \to X$ be the inclusion of a point.*

(1) $H^1_{\text{ét}}(X, i_*(\underline{\mathbb{Z}})) = 0$.
(2) *Assume that X is irreducible and that for each geometric point $\xi\colon \mathrm{Spec}(\Omega) \to X$ the strict henselization X^{sh}_ξ is irreducible.* [44] *Then $H^1_{\text{ét}}(X, \underline{\mathbb{Z}}) = 0$.*

Proof. Consider the distinguished triangle

$$i_*(\underline{\mathbb{Z}}) \to Ri_*(\underline{\mathbb{Z}}) \to Q$$

in $D(X_{\text{ét}}, \underline{\mathbb{Z}})$. Then $Q \in D^{\geq 1}(X_{\text{ét}}, \underline{\mathbb{Z}})$ and thus we obtain an injection

$$0 \to H^1_{\text{ét}}(X, i_*(\underline{\mathbb{Z}})) \to H^1_{\text{ét}}(X, Ri_*(\underline{\mathbb{Z}})) = H^1_{\text{ét}}(\{x\}, \underline{\mathbb{Z}}).$$

By 10.31 the latter group vanishes. This shows (1).

Let $\eta \in X$ be the generic point, and $i\colon \{\eta\} \to X$ the inclusion. We claim that the natural morphism $\underline{\mathbb{Z}} \to i_*(\underline{\mathbb{Z}})$ is an isomorphism. This may be checked on stalks over geometric points $\xi\colon \mathrm{Spec}(\Omega) \to X$. But $X^{\mathrm{sh}}_\xi \times_X \eta$ is exactly the set of generic points of X^{sh}_ξ because X^{sh}_ξ is a cofiltered inverse limit of étale maps over X. As X^{sh}_ξ is irreducible by assumption, this shows that $i_*(\underline{\mathbb{Z}})_\xi \cong \mathbb{Z}$ as desired. Using (1) we can conclude that

$$H^1_{\text{ét}}(X, \underline{\mathbb{Z}}) \cong H^1_{\text{ét}}(X, i_*(\underline{\mathbb{Z}})) = 0$$

as claimed. □

Example 10.34. By 10.27 the same calculation as in 4.29 shows that for the nodal curve $X = \mathrm{Spec}(k[x,y]/(y^2 - x^3 - x^2)) \to \mathrm{Spec}(k)$ over some algebraically closed field k of characteristic $\neq 2$ we have $H^1_{\text{ét}}(X, \underline{\mathbb{Z}}) \cong \mathbb{Z}$. Hence, the assumption on the strict henselizations in 10.33 cannot be dropped.

We can now give an example that the proper base change theorem fails for $\underline{\mathbb{Z}}$, cf. [2, Exposé XII, Section 2].

Example 10.35. Let k be an algebraically closed field, and consider a proper morphism $f\colon Y \to X$ of relative dimension 1 with X a non-singular curve and $Y \to \mathrm{Spec}(k)$ smooth. Assume that $x \in X$ is a closed point, f is smooth over $X \setminus \{x\}$ and that $f^{-1}(x)$ is an irreducible curve with a unique singularity, which is an ordinary double point. Then $H^1_{\text{ét}}(Y, \underline{\mathbb{Z}}) = 0$ by 10.33 because Y is regular, and in particular, geometrically unibranch. Using the same argument for preimages of étale neighborhoods of x, we see that $R^1 f_*(\underline{\mathbb{Z}})_x = 0$. On the other hand, $H^1(f^{-1}(x), \underline{\mathbb{Z}}) \cong \mathbb{Z}$ by 10.34 and thus the proper base change theorem fails.

To get a concrete example, one can take $X = \mathbb{A}^1_k = \mathrm{Spec}(k[t])$ with $\mathrm{char}(k) \neq 2$ and $Y \subseteq \mathbb{P}^2_X$ as the vanishing locus of the homogeneous polynomial $zy^2 - x^3 - x^2 z - tz^3$.

Let us try to understand the $\underline{\mathbb{Z}}$-cohomology of a smooth curve over some algebraically closed field k.

Assume that $j\colon \{\eta\} \to Y$ is the inclusion of the generic point of some (qcqs) integral curve Y over k.

Lemma 10.36. *Let $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(Y_{\text{ét}})$ and assume that $j^*\mathcal{F} = 0$. Then the natural map*

$$\mathcal{F} \to \prod_{y \in Y\, closed} i_{y,*} i_y^* \mathcal{F}$$

yields an isomorphism $\mathcal{F} \cong \bigoplus_{y \in Y\, closed} i_{y,*} i_y^* \mathcal{F}$. In particular, $H^i_{\text{ét}}(Y, \mathcal{F}) = 0$ for $i > 0$ and $H^0_{\text{ét}}(Y, \mathcal{F}) \cong \bigoplus_{y \in Y\, closed} i_y^*(\mathcal{F})$.

Let $\overline{\eta}\colon \mathrm{Spec}(\overline{k(\eta)}) \to Y$ be a geometric point with image $\eta \in Y$. The assertion $j^*\mathcal{F} = 0$ is equivalent to $\mathcal{F}_{\overline{\eta}} = 0$ by 10.8.

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[44] Such a scheme is called "geometrically unibranch", cf. [Stacks, Tag 0CB4], [Stacks, Tag 06DM]. For example, X could be normal, cf. [Stacks, Tag 0BQ3].

Proof. The last assertion follows from the first using 10.9, that $k(y)$ is algebraically closed and 8.1. Let $V \to Y$ be qcqs étale and $s \in \mathcal{F}(V)$ a section. Because $s_{\overline{\eta}} = 0$, there exists an étale morphism $W \to V$ with open, dense image such that $s_{|W} = 0$. This implies that the morphism $\mathcal{F} \to \prod\limits_{y \in Y \text{closed}} i_{y,*}i_y^*\mathcal{F}$ factors over the injection

$$\bigoplus_{y \in Y \text{closed}} i_{y,*}i_y^*\mathcal{F} \to \prod_{y \in Y \text{closed}} i_{y,*}i_y^*\mathcal{F}.$$

That the resulting morphism $\mathcal{F} \to \bigoplus\limits_{y \in Y \text{closed}} i_{y,*}i_y^*\mathcal{F}$ is an isomorphism can be checked on stalks, where it is easy (using 10.9). □

Lemma 10.37. *Assume that $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(Y_{\text{ét}})$ is an étale sheaf of $\mathbb{Q}$-vector spaces. Then $H^i_{\text{ét}}(Y, \mathcal{F}) = 0$ for $i \geq 2$.*

Proof. By 9.15 we can calculate that

$$R^i j_*(j^*\mathcal{F}) = 0$$

for $i \geq 1$ because the stalks are calculated by Galois cohomology of a $\mathbb{Q}$-vector space. We get that

$$H^i_{\text{ét}}(X, j_*(j^*\mathcal{F})) \cong H^i_{\text{ét}}(\{\eta\}, j^*\mathcal{F}) = 0$$

for $i \geq 1$. Let K, Q, H be the kernel resp. image resp. cokernel of $\mathcal{F} \to j_*(j^*\mathcal{F})$. Then

$$j^*K = j^*H = 0$$

and thus K, H have no higher cohomology by 10.36. We can conclude

$$H^i_{\text{ét}}(Y, \mathcal{F}) \cong H^i_{\text{ét}}(Y, Q)$$

for $i \geq 1$, and $H^i_{\text{ét}}(Y, Q) \cong H^i_{\text{ét}}(Y, j_*j^*\mathcal{F})$ for $i \geq 2$. This concludes the proof. □

Set $\mathbb{Z}' := \mathbb{Z}[1/p]$ if $\mathrm{char}(k) = p > 0$ and $\mathbb{Z}' := \mathbb{Z}$ if $\mathrm{char}(k) = 0$.

Lemma 10.38. *Assume that $Y \to \mathrm{Spec}(k)$ is additionally smooth and connected. Then*

$$H^i_{\text{ét}}(Y, \underline{\mathbb{Z}}) \cong \begin{cases} \mathbb{Z}, \ i = 0 \\ 0, \ i = 1 \\ H^{i-1}_{\text{ét}}(Y, \underline{\mathbb{Q}/\mathbb{Z}}), \ i = 2, 3 \\ 0, \ i \geq 4. \end{cases}$$

Moreover,

$$H^i_{\text{ét}}(Y, \underline{\mathbb{Q}/\mathbb{Z}'}) \cong \begin{cases} \mathbb{Q}/\mathbb{Z}', \ i = 0 \\ \mathrm{Pic}(Y)_{\mathrm{tor}}[1/p], \ i = 1 \\ \mathbb{Q}/\mathbb{Z}', \ i = 2 \\ 0, \ i \geq 3. \end{cases}$$

Proof. We can use the short exact sequence

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{Q}} \to \underline{\mathbb{Q}/\mathbb{Z}} \to 0$$

on $Y_{\text{ét}}$. Then the claim follows from 10.33, 10.37 and 9.5. More precisely, we can write $\mathbb{Q}/\mathbb{Z}' = \varinjlim\limits_{n \text{ prime to } p} 1/n\mathbb{Z}/\mathbb{Z}$ and use 8.1 to calculate $\mathbb{Q}/\mathbb{Z}'$-cohomology via 9.5. □

Let us note the following variant for torsion coefficients.

Lemma 10.39. *Let k be a separably closed field and $Y \to \mathrm{Spec}(k)$ a qcqs finite type morphism of dimension ≤ 1. Then $H^i_{\text{ét}}(Y, \mathcal{F}) = 0$ for $i \geq 3$ and any torsion abelian sheaf on $Y_{\text{ét}}$.*

Proof. By 8.3 we may assume that k is algebraically closed. Moreover, we can reduce to Y being integral, e.g., by passing to the normalization. By 9.17, 9.19 and 10.36 we can conclude. □

10.40. **Interlude on formal geometry.** In order to follow the quite illustrative proof of the proper base change theorem in [8, Arcata IV] we need some formal geometry. Thus, we do a short glimpse on formal schemes, without developing their theory from scratch.

Let X be any scheme and $Z \subseteq X$ a closed subscheme defined by some quasi-coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$. Then $\mathcal{I}^{n+1}$ is quasi-coherent for $n \geq 1$, and defines a closed subscheme Z_n of X. Clearly, $Z = Z_0$ and $Z_0 \subseteq Z_n$ is a thickening, i.e., the closed subscheme Z_0 in Z_n is defined by some nilpotent quasi-coherent ideal sheaf. Now intuitively, a formal scheme $\mathfrak{Y}$ is given by some ind-system

$$Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots$$

of thickenings of schemes, and as the most prominent example the formal completion $\widehat{X}_Z$ of X along Z is given by the ind-system

$$Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \ldots .$$

Clearly, we'd expect to have a natural morphism $\alpha_X \colon \widehat{X}_Z \to X$ of formal schemes. In order to realise this heuristic rigorously there are two possible approaches, cf. [Stacks, Tag 0AHY]:

(1) Let $(\mathrm{Sch}) \to \mathrm{Fun}((\mathrm{Sch}), (\mathrm{Sets}))$, $Y \mapsto h_Y$ be the Yoneda embedding. Set

$$\mathfrak{Y} := \varinjlim_n h_{Y_n},$$

where the colimit is taken as fppf- or fpqc-sheaves.

(2) Define $\mathfrak{Y}$ as the locally topologically ringed space with underlying topological space $|\mathfrak{Y}| := |Y_0| \cong |Y_n|$ and structure sheaf

$$\mathcal{O}_{\mathfrak{Y}} := \varprojlim_n \mathcal{O}_{Y_n}.$$

Both approaches have their advantages and disadvantages. For example, the first is quite intuitive, while it is easier to define sheaves and their cohomology via the second approach. Assume now that $X = \mathrm{Spec}(A)$ is affine and $Z = V(I)$ for $I \subseteq A$ some ideal, which we assume finitely generated. In this case the formal completion of X along Z is given by the ind-system

$$\mathrm{Spec}(A/I) \subseteq \mathrm{Spec}(A/I^2) \subseteq \mathrm{Spec}(A/I^3) \subseteq \ldots .$$

This motivates to consider the I-adic completion $\widehat{A}_I := \varprojlim_n A/I^n$, which is an adic ring in the following sense.[45]

Definition 10.41. An adic ring B is a complete topological ring whose topology is J-adic for some ideal $J \subseteq B$.

In both approaches to formal schemes the local building blocks are given by $\mathrm{Spf}(B)$, where B is an adic ring. In the functorial approach, $\mathrm{Spf}(B)$ is defined as the functor

$$R \mapsto \mathrm{Hom}_{\mathrm{cont}}(B, R) = \varinjlim_n \mathrm{Hom}(B/J^n, R)$$

on rings. Here, R is given the discrete topology and the functor extends uniquely to all schemes by the fpqc-sheaf property. In the other approach, one defines the underlying topological space of $\mathrm{Spf}(B)$ as the set of *open* prime ideals of B and equips it with a suitable sheaf $\mathcal{O}_{\mathrm{Spf}(B)}$ of topologial rings, e.g., $\mathcal{O}_{\mathrm{Spf}(B)}(\mathrm{Spf}(B)) = B$. In both cases, one then defines (adic) formal schemes by glueing the (adic) affine formal schemes $\mathrm{Spf}(B)$'s along open formal subschemes. Let us note that

$$\mathrm{Hom}_{\text{formal schemes}}(\mathrm{Spf}(C), \mathrm{Spf}(B)) = \mathrm{Hom}_{\mathrm{cont}}(B, C)$$

in either case. In particular, the category of schemes embeds fully faithfully into the category of formal schemes.

Definition 10.42. A formal scheme $\mathfrak{X}$ is locally noetherian if locally $\mathfrak{X} \cong \mathrm{Spf}(B)$ for B a noetherian adic ring.

If $\mathfrak{X}$ is a locally noetherian scheme, then the structure sheaf $\mathcal{O}_{\mathfrak{X}}$ is coherent and (using the second approach) we get a nice abelian category $\mathrm{Coh}(\mathfrak{X})$ of coherent $\mathcal{O}_{\mathfrak{X}}$-modules.[46] More concretely, if $\mathfrak{X} \cong \mathrm{Spf}(B)$, then the functor

$$\{\text{finitely generated } B\text{-modules}\} \to \mathrm{Coh}(\mathfrak{X}),\ M \mapsto M \otimes_B \mathcal{O}_{\mathfrak{X}}$$

[45] This uses that I is finitely generated.

[46] A priori, one might want to put topologies on the coherent sheaves for which they are complete, but for finitely generated modules over an adic noetherian ring this is not necessary - the adic topology is here already complete and each morphism is strict.

is an equivalence. If $\mathfrak{X} = \widehat{X}_Z$ is the formal completion of X along Z and Z_n the thickenings of Z, then $\mathrm{Coh}(\mathfrak{X})$ is equivalent to inverse systems $\mathcal{F}_n$ with $\mathcal{F}_n$ a coherent $\mathcal{O}_{Z_n}$-module such that the transition map $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induces an isomorphism

$$\mathcal{F}_{n+1} \otimes_{\mathcal{O}_{Z_{n+1}}} \mathcal{O}_{Z_n} \cong \mathcal{F}_n,$$

cf. [Stacks, Tag 087W].

Assume now that A is a noetherian ring and $f \colon X \to S := \mathrm{Spec}(A)$ is a proper morphism. Let $I \subseteq A$ be an ideal and assume that A is I-adically complete. Set

$$S_n := \mathrm{Spec}(A/I^{n+1}),\ X_n := X \times_S S_n$$

for $n \geq 1$. By definition, the formal completion of S along $\mathrm{Spec}(A/I)$ is $\mathrm{Spf}(A)$. Let $\mathfrak{X} = \widehat{X}_Z$ be the formal completion at $Z := \mathrm{Spec}(A/I) \times_S X$. Then we obtain a commutative (in fact cartesian) diagram of formal schemes

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\alpha_X} & X \\ {\scriptstyle \widehat{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spf}(A) & \xrightarrow{\alpha_S} & S. \end{array}$$

The theorem of formal functions and Grothendieck's existence theorem now imply the following "formal GAGA" statement.

Theorem 10.43. (1) *For any $\mathcal{M} \in \mathrm{Coh}(X)$ and $i \geq 0$ the natural map*

$$\Phi \colon \alpha_S^* R^i f_*(\mathcal{M}) \to R^i \widehat{f}_*(\alpha_X^* \mathcal{M})$$

is an isomorphism. In particular,

$$H^i(X, \mathcal{M}) \cong \varprojlim_n H^i(X_n, \mathcal{M}/I^{n+1}\mathcal{M}).$$

(2) *The functor $\alpha_X^* \colon \mathrm{Coh}(X) \to \mathrm{Coh}(\mathfrak{X})$ is an equivalence.*

Proof. Statement (1) is also called the theorem of formal functions and it is proven in [Stacks, Tag 087U]. More precisely, the second assertion implies the first. Namely, $\alpha_S^* \colon \mathrm{Coh}(S) \to \mathrm{Coh}(\mathrm{Spf}(A))$ is an equivalence because any finitely generated A-module is already complete (A is adic noetherian). Thus, it suffices to check Φ is an isomorphism on global sections. Here, the LHS identifies with $H^i(X, \mathcal{M})$ and the RHS with $\varprojlim_n H^i(X_n, \mathcal{M}/I^{n+1}\mathcal{M})$ (this is a property of the coherent cohomology of a proper formal scheme, cf. [12, III.Corollaire 3.4.4]). The fully faithfulness statement in (2) follows from (1), cf. [Stacks, Tag 0883]. Essential surjectivity is proven in [Stacks, Tag 088C] by a reduction to the case of $X = \mathbb{P}^n_A$. □

Let us note the following corollaries.

Theorem 10.44. *With the assumptions of 10.43, the following hold:*

(1) *The map $\pi_0(X_0) \to \pi_0(X)$ is a bijection.*
(2) *The functor $Y \mapsto Y \times_X X_0$ induces an equivalence*

$$\Phi \colon \{Y \to X \text{ finite, étale } \} \to \{Y_0 \to X_0 \text{ finite, étale.}\}$$

Proof. Let us prove statement (1). It suffices to see that the map $\Gamma(X, \mathcal{O}_X) \to \Gamma(X_0, \mathcal{O}_{X_0})$ induces a bijection on idempotents. As $|X_0| = |X_n|$ for any $n \geq 1$, we see that

$$\mathrm{Idem}(\varprojlim_n \Gamma(X_n, \mathcal{O}_{X_n})) \cong \mathrm{Idem}(\Gamma(X_0, \mathcal{O}_{X_0})).$$

By 10.43,

$$\varprojlim_n \Gamma(X_n, \mathcal{O}_{X_n}) \cong \Gamma(X, \mathcal{O}),$$

and the claim follows. As in the proof of 8.3 we can prove that the functor Φ is fully faithful by using 10.47. Indeed, if Y, Y' are finite étale over X, then X-morphisms $Y \to Y'$ identify with open and closed subsets $\Gamma \subseteq Y \times_X Y'$ such that $\Gamma \to Y$ is locally free of rank 1. Now, statement (1) applies as $Y \times_X Y'$ is proper over S. Let now $g_0 \colon Y_0 \to X_0$ be a finite, étale morphism. By 8.3 for any $n \geq 0$ the categories of étale X_n-schemes and étale X_0-schemes are equivalent (via the base change from X_n to X_0). This equivalence preserves finite étale schemes, e.g., by [Stacks, Tag 09ZV] or using Zariski's main theorem 10.22 to see that qcqs separated, universally closed étale morphisms are finite. This implies that Y_0 lifts uniquely to a finite étale scheme $g_n \colon Y_n \to X_n$. The inverse system $g_{n,*}\mathcal{O}_{Y_n}$ defines now a coherent module $\mathcal{A}$ on the formal completion $\mathfrak{X}$ of X, in fact a finite, locally free $\mathcal{O}_{\mathfrak{X}}$-algebra. By Grothendieck's existence theorem 10.43 $\mathcal{A} \cong \alpha_X^* \mathcal{B}$ for some finite, locally free $\mathcal{O}_X$-algebra $\mathcal{B}$ and then $Y := \underline{\mathrm{Spec}}_X(\mathcal{B})$ defines an X-scheme such that $Y \times_X X_0 \cong Y_0$. This implies that $Y \to X$ is finite étale and thus the claim. □

10.45. **First cases of the proper base change theorem.** By 10.8 and 10.11 the proper base change theorem 10.1 is equivalent to the following theorem.

Theorem 10.46. *Let A be a strictly henselian local ring, $S := \mathrm{Spec}(A)$ and $s \in S$ the unique closed point. Let $f\colon X \to S$ be a proper morphism and $\mathcal{F}$ a torsion abelian sheaf on $X_{\text{ét}}$. Let $i\colon X_s := X \times_S s \to X$ be the inclusion of the special fiber. Then the natural map*

$$H^i_{\text{ét}}(X, \mathcal{F}) \to H^i_{\text{ét}}(X_s, i^*\mathcal{F})$$

is an isomorphism for any $i \geq 0$.

Using approximation techniques (for X, A and $\mathcal{F}$) one reduces to the case that A is noetherian, and even the strict henselization of a finite type $\mathbb{Z}$-algebra. In this case, the ring A is excellent. For the details of this reduction, we refer to the literature, e.g., [2, Exposé XII]. Instead of following [Stacks, Tag 095S], we will follow [8, Arcata IV] as the arguments there are quite illustrative.

We first want to explain the case of 10.46 when $\mathcal{F} = \underline{\Lambda}$ is a constant sheaf (with Λ some torsion abelian group, when necessary), and $i = 0, 1$.

Note that

$$H^0_{\text{ét}}(X, \underline{\Lambda}) \cong \mathrm{Hom}_{\mathrm{cont}}(\pi_0(X), \Lambda)$$

for the connected components $\pi_0(X)$ of X.

Hence, the special case $i = 0$ and $\mathcal{F} = \underline{\Lambda}$ of 10.46 follow from the following theorem.

Theorem 10.47. *Let A be a local henselian, noetherian ring, $S := \mathrm{Spec}(A)$ with closed point s and $f\colon X \to S$ a proper morphism. Then the map*

$$\pi_0(X_0) \to \pi_0(X)$$

is a bijection, where $X_0 = X_s = X \times_S s$ denotes the special fiber.

Proof. Let $A^\wedge$ be the completion of A and $X' := X \times_S \mathrm{Spec}(A^\wedge)$ By 10.44 we know that

$$\pi_0(X_0) \cong \pi_0(X')$$

(note that $A^\wedge/I \cong A/I$ and hence $X' \times_S \mathrm{Spec}(A/I) \cong X_0$). The properness of f implies that $B := \Gamma(X, \mathcal{O})$ is a finite A-algebra. Flat base change implies that

$$B \otimes_A A^\wedge \cong \Gamma(X', \mathcal{O}_{X'}).$$

As A is noetherian and B finite over A we see that $B \otimes_A A^\wedge$ is the $\mathfrak{m}_A$-adic completion $B^\wedge$ of B. As A is local henselian the finiteness of B over A implies

$$\mathrm{Idem}(B) \cong \mathrm{Idem}(B/\mathfrak{m}B)$$

by 10.16. This implies that

$$\mathrm{Idem}(B) \cong \mathrm{Idem}(B/\mathfrak{m}B) \cong \mathrm{Idem}(B^\wedge) \cong \mathrm{Idem}(\Gamma(X', \mathcal{O}_{X'}))$$

as desired. (As a warning let us mention that $B/\mathfrak{m}B$ need not be isomorphic to $\Gamma(X_0, \mathcal{O}_{X_0})$). □

Next we discuss the case that $i = 1$ and $\mathcal{F} = \underline{\mathbb{Z}/n}$ for some $n \in \mathbb{Z} \setminus \{0\}$ of 10.46.

Theorem 10.48. *Let A be a local henselian, noetherian ring, $S := \mathrm{Spec}(A)$ and $s \in S$ the closed point. Let $f\colon X \to S$ be a proper morphism and $X_0 := X \times_S s$ the special fiber. Then the functor*

$$\Phi\colon \{Y \to X \text{ finite, étale}\} \to \{Y_0 \to X_0 \text{ finite, étale}\},\ Y \mapsto Y \times_X X_0$$

is an equivalence of categories. In particular, for any finite group G the map

$$H^1_{\text{ét}}(X, \underline{G}) \to H^1_{\text{ét}}(X_0, \underline{G})$$

is a bijection.

Proof. The last assertion follows from the first because for G a finite group each G-torsor over a scheme Z is represented by some finite étale Z-scheme. As in the proof of 8.3 we can prove that the functor Φ is fully faithful by using 10.47. Indeed, if Y, Y' are finite étale over X, then X-morphisms $Y \to Y'$ identify with open and closed subsets $\Gamma \subseteq Y \times_X Y'$ such that $\Gamma \to Y$ is surjective. Now, 10.47 applies as $Y \times_X Y'$ is proper over S. Let as in the proof of 10.47 $A^\wedge$ be the completion of A at its maximal ideal $\mathfrak{m}_A$, and $X' := X \times_S \mathrm{Spec}(A^\wedge)$. By 10.44 we see that it suffices to see that for each finite étale X'-scheme Y' there exists a finite étale X-scheme Y such that $Y' \times_{X'} X_0 \cong Y \times_X X_0$. Using approximation by finite type schemes over $\mathbb{Z}$, we may assume that A is a local henselian, excellent ring. Now, consider the functor

$$B \mapsto \{\text{isomorphism classes of finite étale schemes over } X \times_S \mathrm{Spec}(B)\}.$$

Then Artin approximation 10.49 applied to it implies the existence of Y. This finishes the proof. □

Here is the statement of Artin approximation that we used, cf. [5, Lemma 2.1.3].

Theorem 10.49. *Let A be a local henselian, excellent ring with completion $A^\wedge$. Let $F\colon (\mathrm{Alg}_A) \to (\mathrm{Sets})$ be a functor, which sends filtered colimits of A-algebras to filtered colimits. Then for any element $\xi \in F(A^\wedge)$ and any $n \geq 1$ there exists an element $\eta \in F(A)$ such $\xi = \eta \in F(A/\mathfrak{m}^n)$.*

Proof. The essential ingredient is Popescu's theorem, cf. [Stacks, Tag 07GC]: any flat morphism $R \to R'$ of noetherian rings, whose geometric fibers are regular, is a filtered colimits of *smooth* ring maps with source R.

By excellency of A this can be applied to $A \to A^\wedge$, i.e., $A^\wedge = \varinjlim_{i\in I} A_i$ for some filtered system of smooth maps $A \to A_i$. Now, let $\xi \in F(A^\wedge)$. By assumption on F there exists some $i \in I$, such that ξ is the image of some $\xi_i \in F(A_i)$ along $F(A_i) \to F(A^\wedge)$. We know that $A/\mathfrak{m}^n \cong A^\wedge/\mathfrak{m}^n$. As A is henselian, the resulting morphism $A_i \to A/\mathfrak{m}^n$ lifts to a section $A_i \to A$ by 10.50. Now we can conclude because the image η of ξ_i along $F(A_i) \to F(A)$ does the job. $\square$

Lemma 10.50. *Let A be a local henselian ring, B a smooth A-algebra and $n \geq 1$. Then any morphism $B \to A/\mathfrak{m}_A^n$ of A-algebras lifts to a morphism $B \to A$ of A-algebras.*

Proof. For simplicity, we assume $n = 1$ (this case is sufficient for us to prove 10.48). The general case is proven in [16, p. I.8] (even for general henselian pairs). Set $k := A/\mathfrak{m}A$ and let $x \in \mathrm{Spec}(B)$ be the image of the section $\mathrm{Spec}(k) \to \mathrm{Spec}(B)$. As $A \to B$ is smooth there exists an affine open neighborhood U of x and an étale morphism $U \to \mathrm{Spec}(A[T_1, \ldots, T_m])$ such that x maps to the zero section. Now, we may replace B by $B \otimes_{A[T_1,\ldots,T_m]} A$, where $T_i \mapsto 0 \in A$, and reduce to the case that $A \to B$ is étale. Then the existence of the lift $B \to A$ is one characterization of a local henselian ring, cf. 10.16. $\square$

10.51. **The constructible topology.** Let X be a qcqs scheme. Then the underlying topological space of X is spectral, i.e., it is quasi-compact, each closed irreducible subset has a unique generic point and there exists a basis of quasi-compact open subsets, which is stable under finite intersections, cf. [Stacks, Tag 08YG]. Now each spectral space admits a certain finer topology, namely the constructible topology.

Definition 10.52 ([Stacks, Tag 08YF])**.** Let S be a spectral space. Then S_{cons} is the topological space with underlying set S and the coarsest topology on S such that for each quasi-compact open $U \subseteq S$ the sets U and $S \setminus U$ are open.

Clearly, the identity is a continuous map $S_{\mathrm{cons}} \to S$.

Theorem 10.53. *Let S be a spectral space. Then S_{cons} is profinite, i.e., compact Hausdorff and totally disconnected.*

Proof. This is [Stacks, Tag 0901]. Admitting that each spectral space S is a cofiltered inverse limit $\varprojlim_{i\in I} S_i$ of finite T_0-spaces ([Stacks, Tag 09XX]) a slick proof can be given as follows: By definition of the inverse limit topology a *quasi-compact* open subset of S is exactly the inverse image of an open subset of some S_i. This implies that the constructible topology on S is the inverse limit topology when each S_i is given the discrete topology. Indeed, the constructible topology on a finite T_0-space is the discrete topology. But this implies that S_{cons} is profinite. $\square$

Example 10.54. $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cons}}$ identifies with the one-point compactification of $\mathbb{N}$.

A continuous map $S' \to S$ of spectral spaces is continuous for the constructible topology if and only if it is spectral, i.e., quasi-compact.

Definition 10.55. Let S be a spectral space. A subset $U \subseteq S$ is called

(1) ind-constructible if U is open in S_{cons},
(2) pro-constructible if U is closed in S_{cons},
(3) constructible if U is open and closed in S_{cons}.

Thus, a subset $U \subseteq S$ is constructible if and only if it is a finite union of $V \cap S \setminus W$ for $V, W \subseteq S$ quasi-compact opens.

Lemma 10.56 ([Stacks, Tag 0903])**.** *Let S be a spectral space and $U \subseteq S$ a subset.*

(1) *If U is pro-constructible and stable under specializations, then U is closed in S.*
(2) *If U is ind-constructible and stable under generalizations, then U is open in S.*

Proof. Passing to complements it suffices to show (1). Let $x \in \overline{U}$. Let $\mathcal{B}$ be the basis of quasi-compact open neighborhoods of x. Recall that $\bigcap\limits_{V \in \mathcal{B}} V$ is the set of generalizations of x. We claim that

$$\bigcap_{V \in \mathcal{B}} V \cap U \neq \emptyset.$$

Now, $V, U \subseteq S_{\mathrm{cons}}$ for $V \in \mathcal{B}$ are closed and hence each $V \cap U$ is a compact, Hausdorff space by Theorem 10.53. As each $V \cap U$ is non-empty (as $x \in \overline{U}$) their intersection is therefore non-empty, cf. [Stacks, Tag 0A2R]. This finishes the proof. □

The following is Chevalley's theorem.

Theorem 10.57. *Let $f \colon Y \to X$ be a morphism of qcqs schemes, which is of finite presentation. Then $f_{\mathrm{cons}} \colon Y_{\mathrm{cons}} \to X_{\mathrm{cons}}$ is open and closed.*

Using 10.56 and 10.57 we can conclude that each generalizing quasi-compact morphism of schemes, which is locally of finite presentation is open. This proves 5.24.

Proof. The closedness is easy. Namely, if $S' \to S$ is any spectral map of spectral spaces, then $S'_{\mathrm{cons}} \to S_{\mathrm{cons}}$ is a continuous map of compact Hausdorff spaces and hence closed. The openess is proven in [Stacks, Tag 054K]. □

One motivation for introducing constructible subsets is approximation. If $X = \varprojlim\limits_{i \in I} X_i$ is a cofiltered inverse limit of qcqs schemes X_i along affine transition maps, then each constructible subset of X is the pullback of some constructible subset of some X_i.

For a noetherian topological space S each open set is quasi-compact, and hence the constructible subsets of S are exactly the finite unions of locally closed subsets. In this case one can give the following characterizations of constructible subsets, which we already used in 3.38.

Lemma 10.58. *Let S be a noetherian topological space. Then a subset $U \subseteq S$ is constructible if and only if for any closed irreducible subset $Z \subseteq S$ the intersection $Z \cap U$ is either not dense in Z or contains a non-empty open subset of Z.*

Thus, if U is constructible, then $U \cap Z$ cannot just be the generic point of Z.

Proof. This is [Stacks, Tag 053Z]. □

10.59. **Constructible sheaves.** Let X be a scheme.

Definition 10.60 ([Stacks, Tag 05BE]).
(1) A sheaf of abelian groups $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$ is called finite locally constant if it is represented by some finite étale X-scheme, or equivalently (by descent, 7.13) that there exists an étale covering $\{X_i \to X\}_{i \in I}$ such that $\mathcal{F}_{|X_i} \cong \underline{M}$ for some finite abelian group M.
(2) A sheaf of abelian groups $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$ is called constructible if for any affine open $U \subseteq X$ there exists a finite decomposition $U = \coprod\limits_{i=1}^{n} U_i$ of U into *constructible* locally closed subschemes, such that $\mathcal{F}_{|U_i}$ is finite locally constant.

We denote the full subcategory of $\mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$ of constructible sheaves by $\mathrm{Sh}_c(X_{\text{ét}})$.

Note that by 8.3 the U_i can without loss of generality be assumed to be reduced. We will mostly be interested in qcqs schemes. Here, the decomposition into constructible locally closed subschemes exists globally.

Lemma 10.61 ([Stacks, Tag 095E]). *Assume X is qcqs. Let $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$. The following are equivalent:*

(1) *$\mathcal{F}$ is constructible,*
(2) *there exists an open covering $X = \bigcup\limits_{i \in I} U_i$ such that $\mathcal{F}_{|U_i}$ is constructible,*
(3) *there exists a partition $X = \coprod\limits_{i=1}^{n} X_i$ into constructible locally closed subschemes, such that $\mathcal{F}_{|X_i}$ is finite locally constant for each i.*

Proof. Clearly, (1) implies (2). Assume (2). Then there exists a finite covering $X = \bigcup\limits_{j=1}^{m} V_j$ with $V_j \subseteq X$ affine open and decompositions $V_j = \coprod\limits_{k} V_{j,k}$ into constructible locally closed subschemes, such that $\mathcal{F}_{|V_{j,k}}$ are finite locally constant. Now each $V_{j,k}$ is constructible in X because $V_{j,k,\mathrm{cons}} \subseteq V_{j,\mathrm{cons}} \subseteq X_{\mathrm{cons}}$ are open and closed. Now 10.62 applies and yields the desired X_i.

Now assume (3). If $U \subseteq X$ is open, then we can take $U_i := U \cap X_i$. □

Lemma 10.62 ([Stacks, Tag 09Y4]). *Let T be a spectral space and $T = \bigcup_{j=1}^{m} V_j$ with $V_j \subseteq T$ constructible. Then there exists a finite constructible decomposition $T = \coprod_{i=1}^{n} T_i$ with $T_i \subseteq T$ constructible and locally closed such that each V_j is a union of T_i's.*

Proof. For a subset $Z \subseteq T$ we set $Z^c := T \setminus Z$. After refining the V_j we may assume that $V_j = U_j \cap W_j^c$ for some quasi-compact open subsets $U_j, W_j \subseteq T$. Let I be the finite set of closed subsets of T consisting of $\emptyset, T, U_j^c, W_j^c$ for $j = 1, \ldots, m$ and finite intersections of these. Then each $Z \in I$ is constructible and closed in T. For $Z \in I$ set

$$T_Z := Z \setminus \bigcup_{Z' \in I,\ Z' \subsetneq Z} Z'.$$

Then $T_Z \subseteq T$ is constructible and locally closed, and $T = \coprod_{Z \in I} T_Z$. Let now $t \in T$ and set

$$Z_t := \bigcap_{Z \in I,\ t \in Z} Z.$$

Assume $t \in V_j = U_j \cap W_j^c$ for some $j = 1, \ldots, m$. We claim that $T_{Z_t} \subseteq V_j$, which will finish the proof. Note that $Z_t \subseteq W_j^c$ because $t \in W_j^c$. Assume now that there exists some $z \in T_{Z_t} \setminus V_j$. Then $z \in U_j^c$ (because $z \in T_{Z_t} \subseteq W_j^c$). By definition of T_{Z_t} we get that $t \in U_j^c$, which is a contradiction. □

One motivation for introducing constructible sheaves is approximation. If $X = \varprojlim_{i \in I} X_i$ is a cofiltered inverse limit of qcqs schemes along affine transition maps, then by the third characterization in 10.61 each $\mathcal{F} \in \mathrm{Sh}_c(X_{\text{ét}})$ is the pullback of some constructible sheaf $\mathcal{F}_i$ on some X_i. Together with 8.1, 8.2 and 10.65 this allows approximation in the proof of the proper base change theorem by noetherian schemes, even of finite type over $\mathbb{Z}$.

We can draw further consequences.

Lemma 10.63. *Let X be a scheme.*

1. *If $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$, then $\mathcal{F}$ being constructible can be tested Zariski-locally on X.*
2. *The category of finite locally constant $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$ is closed under finite limits, finite colimits and extensions. In particular, it is abelian.*
3. *The category $\mathrm{Sh}_c(X_{\text{ét}}) \subseteq \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$ is closed under finite limits, finite colimits and extensions. In particular, it is abelian.*
4. *If $f \colon Y \to X$ is a morphism of schemes, then $f^{-1}(\mathrm{Sh}_c(X_{\text{ét}})) \subseteq \mathrm{Sh}_c(Y_{\text{ét}})$.*

Proof. The first part follow from the second assertion of 10.61. The second assertion holds for finite locally constant abelian sheaves on any topos. The crucial statement is each morphism of finite locally constant sheaves is locally constant, and hence the kernel/cokernel are locally constant, too. By (1) the last assertions are local on X and hence we may assume that X, Y are qcqs. Now note that two finite decompositions $X = \coprod_{i=1}^{n} X_i = \coprod_j Y_j$ into constructible locally closed subschemes admit the common refinement $X = \coprod_{i,j} X_i \cap Y_j$. This implies that (3) follows from (2). As the base change of a constructible locally closed decomposition is again a constructible locally closed decomposition, (4) holds. □

The typical example of a constructible sheaf is the following.

Lemma 10.64. *Let $f \colon U \to X$ be an étale morphism of qcqs schemes, and $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(U_{\text{ét}})$ finite locally constant. Then $f_!\mathcal{F}$ is constructible on X.*

Proof. If f is finite étale, then $f_!\mathcal{F} = f_*(\mathcal{F})$ (e.g., by checking on stalks using 10.27) and $f_*(\mathcal{F})$ is finite locally constant on X. Indeed, this claim is étale local on X and hence one can reduce to the case that $U = \coprod_{i=1}^{n} X$ is just a disjoint union of copies of X.

We now reduce the general case to the finite étale case by passing to a stratification on X. As the claim is Zariski-local on X we may assume that X is affine. By approximation we may assume that X is noetherian. By noetherian induction it suffices to find some open, dense subset $V \subseteq X$ such that $U \times_X V \to V$ is finite étale, as then we can refine the decomposition $X = V \coprod X \setminus V$ further using induction. But over the (finitely many) generic points of X, the morphism f is finite étale (using 8.3 to assume that X is reduced). Spreading out, this shows the existence of V. □

Note that each constructibe sheaf is a torsion sheaf. Conversely, we get the following.

Lemma 10.65. *Assume that X is qcqs. Let $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$ be a torsion sheaf. Then $\mathcal{F}$ is a filtered colimit of constructible sheaves.*

Proof. As $\mathcal{F}$ is torsion, $\mathcal{F} \cong \varinjlim_{n \text{ non-zero}} \mathcal{F}[n]$, and we may assume that $\mathcal{F}$ is a sheaf of $\mathbb{Z}/n$-modules for some non-zero $n \in \mathbb{Z}$. Using that qcqs étale morphisms $U \to X$ form a basis for $X_{\text{ét}}$ and 5.18 we can write $\mathcal{F}$ as a cokernel of a map

$$\bigoplus_{i \in I} f_{i,!}(\mathbb{Z}/n) \to \bigoplus_{j \in J} g_{j,!}(\mathbb{Z}/n),$$

where $f_i \colon U_i \to X$, $g_j \colon V_j \to X$ are qcqs étale morphisms and I, J sets. Taking a filtered colimit over finite subsets of I, J, then shows the claim by using 10.64 and 10.63. □

Another motivation for introducting constructible sheaves is a reduction to constant coefficients. The precise statement that we need is the following.

Theorem 10.66. *Let X be a scheme.*

(1) *If $f \colon Y \to X$ is a finite morphism of finite presentation and $\mathcal{F} \in \mathrm{Sh}_c(Y_{\text{ét}})$, then $f_*(\mathcal{F})$ is constructible.*
(2) *Let $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$. If $\mathcal{F}$ is constructible and X qcqs, then there exists finite and finitely presented morphisms $f_j \colon Y_j \to X, i = 1, \ldots, m$, finite abelian groups M_j and an injection*

$$\mathcal{F} \hookrightarrow \prod_{j=1}^{m} f_{j,*}(\underline{M}_i).$$

If X is noetherian, the converse holds.

In order to be able to prove this theorem, we need some preparations.

Lemma 10.67. *Let X be a noetherian scheme and $\mathcal{F} \in \mathrm{Sh}_c(X_{\text{ét}})$. If $\mathcal{G} \subseteq \mathcal{F}$ is a subsheaf, then $\mathcal{G}$ is constructible.*

Proof. Using noetherian induction, we can reduce to the case that X is integral and $\mathcal{F}$ finite locally constant. It suffices to show that $\mathcal{G}_{|U}$ is finite locally constant for some open, dense subset U. To show this we may étale localize on X and hence assume that $\mathcal{F}$ is constant. Let η be the generic point of X and $\overline{\eta}$ be a geometric point of X with image η. Now, after replacing X by some étale neighborhood of $\overline{\eta}$ the stalk $\mathcal{G}_{\overline{\eta}} \subseteq \mathcal{F}_{\overline{\eta}}$ can be spread out to some finite constant subsheaf $\mathcal{F}' \subseteq \mathcal{G}$ with the same stalk at $\overline{\eta}$. Evaluating on connected $V \to X$ étale shows that $\mathcal{F}' = \mathcal{G}$ as subsheaves of the constant sheaf $\mathcal{F}$. □

Lemma 10.68. *Let X be a Noetherian scheme, $\mathcal{F} \in \mathrm{Sh}_c(X_{\text{ét}})$. Assume that $\mathcal{F}_i \subseteq \mathcal{F}, i \in I$, is a filtered system of subsheaves of $\mathcal{F}$. If $\varinjlim_{i \in I} \mathcal{F}_i = \mathcal{F}$, then $\mathcal{F}_i = \mathcal{F}$ for $i \gg 0$.*

Proof. Fix $i_0 \in I$ and replace I by the filtered category i_0/I. Argueing on a decomposition we may assume that $\mathcal{F}$ and $\mathcal{F}_{i_0}$ are locally constant (for $\mathcal{F}_{i_0}$ we use 10.67 here). Enlarging i_0 we may then even assume that $\mathcal{F}_{i_0} = \mathcal{F}$ because checking whether a morphism of locally constant sheaves is an isomorphism can be done at a point for each of the finitely many connected components of X. This implies the claim. □

Lemma 10.69. *Let X be a qcqs scheme and $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$ be a torsion sheaf. Then $\mathcal{F}$ is constructible if and only if the functor $\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})}(\mathcal{F}, -)$ commutes with filtered colimits.*

Proof. If $\mathcal{F}$ is constructible, then $\mathcal{F}$ is the cokernel of a morphism

$$\bigoplus_{i=1}^{n} f_{i,!}(\mathbb{Z}/m_i) \to \bigoplus_{j=1}^{m} f_{j,!}(\mathbb{Z}/m_j)$$

for some $f_i \colon Y_i \to X$ qcqs étale morphisms and $m_i \in \mathbb{Z} \setminus \{0\}$. Indeed, this claim is Zariski local on X and then can by approximation by shown for X noetherian. Then it follows from 10.68 and the proof of 10.65. Now, each $\mathrm{Hom}(f_{i,!}(\mathbb{Z}/m), -) \cong \mathrm{Hom}(\mathbb{Z}/m, f_i^*(-))$ commutes with filtered colimits by 8.1 as the Y_i are qcqs. This implies the same for $\mathcal{F}$ by the 5-lemma and exactness of filtered colimits. Conversely, assume that $\mathrm{Hom}(\mathcal{F}, -)$ commutes with filtered colimits. By 10.65, $\mathcal{F}$ is the filtered colimit of its constructible subsheaves $\mathcal{F}_i$. But then the identify $\mathcal{F} \to \mathcal{F}$ must factor over some $\mathcal{F}_i$ and $\mathcal{F} = \mathcal{F}_i$ is constructible. □

We can now prove 10.66.

Proof of 10.66. We show (1). By Zariski localization and approximation, we may assume that X is noetherian. By noetherian induction it suffices to show that there exists a non-empty open dense subset $U \subseteq X$ such that $f_*(\mathcal{F})_{|U}$ is constructible. Note that the statement is invariant under universal homeomorphisms by 8.3 and that universal homeomorphisms of finite presentation spread out.[47] Generically, $Y \to X$ is a finite étale morphism (up to universal homeomorphisms). This reduces the statement to 10.64.

Now we prove (2). By absolute approximation we may assume that X is noetherian, cf. [Stacks, Tag 01ZA]. Assume that $\mathcal{F}$ is constructible. By 10.68 it suffices to show that for any geometric point $\overline{x}$ of X with image x, there exists a finite morphism $f_*\colon Y \to X$, a finite abelian group M and a morphism $\mathcal{F} \to f_*\underline{M}$ whose stalk at $\overline{x}$ is injective. We may assume that $x \in X$ is the generic point by replacing X by the closure of x. As $\mathcal{F}$ is constructible, there exists a finite extension $K/k(x)$ such that $\mathcal{F}_{|\mathrm{Spec}(K)} \cong \underline{M}$ is constant. Let $f\colon Y \to X$ be the normalization of X in $\mathrm{Spec}(K)$ (note that this morphism is integral, but not necessarily finite - we'll take care of that in a second).

Let $j\colon \mathrm{Spec}(K) \to Y$ be the inclusion. Then $j_*(\underline{M}) \cong \underline{M}$ by normality of Y, cf. 10.33. Consider the composition

$$\mathcal{F} \to f_*(f^*\mathcal{F}) \to f_*(j_*(j^*f^*\mathcal{F}) \cong f_*(j_*(\underline{M})) \cong f_*(\underline{M}).$$

By 10.27 we can conclude that $\mathcal{F}_{\overline{x}} \to (f_*(\underline{M}))_{\overline{x}}$ is injective.

Now write $Y = \varprojlim_{i\in I} Y_i \to X$ as a cofilterd limit of finite morphisms $f_i\colon Y \to X$, cf. [Stacks, Tag 0817]. Then $f_*(\underline{M}) \cong \varprojlim_{i\in I} f_{i,*}(\underline{M})$ by 8.2. Now by 10.69 the morphism $\mathcal{F} \to f_*(\underline{E})$ will factor over some $f_{i,*}(\underline{E})$ as desired. Finally, 10.67 and (1) the converse holds. □

10.70. **Reduction to constant coefficients.** We now explain how to reduce the proper base change theorem to constant coefficients.

Proposition 10.71. *Let X be a noetherian scheme and Z a closed subscheme. Assume that for any finite morphism $X' \to X$, any $n \in \mathbb{Z}\setminus\{0\}$ the map*

$$H^i_{\text{ét}}(X', \mathbb{Z}/n) \to H^i_{\text{ét}}(Z \times_X X', \mathbb{Z}/n)$$

is bijective for $i = 0$ and surjective for $i > 0$. Then for any torsion sheaf $\mathcal{F}$ on X the map

$$H^i_{\text{ét}}(X, \mathcal{F}) \to H^i_{\text{ét}}(Z, \mathcal{F}_{|Z})$$

is bijective for any $i \geq 0$.

In order to prove this proposition, we need to investigate some homological algebra.

Definition 10.72. Let $\mathcal{C}$ be an abelian category, and $T\colon \mathcal{C} \to (\mathrm{Ab})$ be a functor. The functor T is called effacable if for any $A \in \mathcal{C}$ and any $\alpha \in T(A)$ there exists a monomorphims $u\colon A \to M$ such that $T(u)(\alpha) = 0$.

The basic example is the following.

Lemma 10.73. *Let X be a qcqs scheme and $\mathcal{C} = \mathrm{Sh}_c(X_{\text{ét}})$. Then $T(-) := H^i_{\text{ét}}(X, -)$ is effacable for any $i > 0$.*

Proof. Let $\mathcal{F}$ be a constructible sheaf. Then $n\mathcal{F} = 0$ for some $n \geq 1$. Let now $\mathcal{F} \to \mathcal{I}$ be an injection with $\mathcal{I}$ an injective sheaf of $\mathbb{Z}/n$-modules. Then $H^i_{\text{ét}}(X, \mathcal{I}) = 0$ and by 10.65 $\mathcal{I}$ is a colimit of its constructible subsheaves. By 10.69 and 8.1 we can conclude. □

Now, by 10.66 the following lemma implies 10.71 by taking

$$T^\bullet(-) = H^\bullet_{\text{ét}}(X, -),\ S^\bullet(-) = H^\bullet(Z, (-)_{|Z}),\ \mathcal{C} = \mathrm{Sh}_c(X_{\text{ét}})$$

and $\mathcal{E}$ as the class of constructible sheaves of the form $\prod_{i=1}^n f_{i,*}(\underline{M_i})$, where $f_i\colon X_i \to X$ is a finite morphism and M_i a finite abelian group.

Lemma 10.74. *Let $\mathcal{C}$ be an abelian category and $\varphi^\bullet\colon T^\bullet \to S^\bullet$ be a natural transformation of cohomological δ-functors (with values in abelian groups). Assume that T^i is effacable for $i > 0$ and let $\mathcal{E} \subseteq \mathcal{C}$ a class of objects such that each $A \in \mathcal{C}$ can be embedded into some $E \in \mathcal{E}$. Then the following properties are equivalent:*

(1) *$\varphi^i_A\colon T^i(A) \to S^i(A)$ is bijective for any $i \geq 0$ and $A \in \mathcal{C}$.*
(2) *For $E \in \mathcal{E}$ the map φ^0_E is bijective, and φ^i_E is surjective for $i > 0$.*

[47] The critical point is spreading out the surjectivity of the diagonal, cf. [Stacks, Tag 07RR], which implies universal injectivity.

Proof. The implication (1) $\Rightarrow$ (2) is trivial. Let us assume (2). We argue by induction on $i \geq 0$. Let $A \in \mathcal{C}$. Let $A \to E$ be an injection with $E \in \mathcal{E}$. As $T^0(A) \to T^0(E), S^0(A) \to S^0(E)$ are injective, we see that φ_A^0 is injective. The injectivity of $\varphi_{A/E}^0$ then implies bijectivity of φ_A^0 by the 5-lemma. Now assume that statement is known for some $i \geq 0$. Consider the commutative diagram

$$\begin{CD}
T^i(E) @>>> T^i(E/A) @>>> T^{i+1}(A) @>>> T^{i+1}(E) @>>> T^{i+1}(E/A) \\
@VV{\cong}V @VV{\cong}V @VV{\varphi_A^{i+1}}V @VV{\varphi_E^{i+1}}V @VV{\varphi_{E/A}^{i+1}}V \\
S^i(E) @>>> S^i(E/A) @>>> S^{i+1}(A) @>>> S^{i+1}(E) @>>> S^{i+1}(E/A)
\end{CD}$$

with exact rows. Let $a \in T^{i+1}(A)$ with $\varphi_A^{i+1}(a) = 0 \in S^{i+1}(A)$. By effacability of T^{i+1} we may, after possibly changing E, assume that a maps to 0 in $T^{i+1}(E)$. Then a diagram chase reveals that $a = 0$. Using that φ_E^{i+1} is surjective and $\varphi_{E/A}^{i+1}$ injective (this we have shown already for any object in $\mathcal{C}$) a diagram chase shows that φ_A^{i+1} is surjective. □

We can draw the following corollary.

Lemma 10.75. *Proper base change holds in cohomological degree* 0 *for any proper morphisms and for any torsion abelian sheaf.*

Proof. Using the usual approximations the proofs of 10.74 and 10.71 reduce this to 10.47. □

10.76. **End of the proof.** We now finish the proof of 10.1.

Consider a commutative diagram

$$\begin{CD}
Y' @>{g''}>> Y \\
@V{h'}VV @VV{h}V \\
X' @>{g'}>> X \\
@V{f'}VV @VV{f}V \\
S' @>{g}>> S
\end{CD}$$

with both squares cartesian. If 10.1 holds for f and h, i.e., for the two inner squares, then it holds also for $f \circ h$, i.e., the outer (cartesian) square. This can be checked directly.

Lemma 10.77. *The following are equivalent:*

(1) *Proper base change holds for f and any torsion abelian étale sheaf on X,*
(2) *For any prime ℓ, any injective sheaf $\mathcal{F}$ of $\mathbb{F}_\ell$-modules on $X_{\text{ét}}$ and any $q > 0$ the sheaf $R^q f'_*(g'^{,*}\mathcal{F})$ on S' is trivial*

Proof. That 1) implies 2) is trivial. The reverse implication is a usual dévissage: First one reduces to sheaves of $\mathbb{F}_\ell$-modules. Then one can use 10.74. □

Lemma 10.78. *Assume that proper base change holds for h and $f \circ h$, and that h is surjective. Then proper base change holds for f.*

Proof. We use 10.77 and pick some injective $\mathbb{F}_\ell$-sheaf $\mathcal{I}$ on $X_{\text{ét}}$. We have to show that $R^q f_*(g'^{,*}\mathcal{I}) = 0$ for $q > 0$. Let $h^*\mathcal{I} \to \mathcal{J}$ be an injection with $\mathcal{J}$ an injective $\mathbb{F}_\ell$-sheaf on Y. Then the adjoint map $\mathcal{I} \to h_*(\mathcal{J})(= Rh_*(\mathcal{J}))$ is injective as can be checked on stalks using that h is surjective. As $\mathcal{I}$ is injective, this injection split and it suffices to show the claim for $\mathcal{I}$ replaced by the injective $\mathbb{F}_\ell$-sheaf $h_*(\mathcal{J})$. Now, our assumptions imply

$$Rf'_*(g'^{,*}h_*(\mathcal{J})) \cong Rf'_*(Rh'_*(g''^{,*}\mathcal{J})) = R(f' \circ h')_*(g''^{,*}(\mathcal{J})) = g^*R(f \circ h)_*(\mathcal{F}) \in \mathcal{D}^{\leq 0}$$

as desired. □

Checking on stalks, reduces the proof of proper base change for f to the case that S is affine. By Chow's lemma there exists a diagram

$$\begin{array}{ccc}
Y & \xrightarrow{\ h\ } & X \\
& {\scriptstyle f\circ h}\searrow & \downarrow{\scriptstyle f} \\
& & S
\end{array}$$

with h projective, surjective, and $f \circ h$ projective. By 10.78 we can thus assume that f is projective. Picking some closed immersion $i\colon X \to \mathbb{P}^n_S \to S$ reduces to the case that $X = \mathbb{P}^n_S$ (using that the proper base change theorem holds for closed immersions by 10.9).

Lemma 10.79. *Let A be any ring and let $F_0, \ldots, F_n \in A[x_1, y_1, \ldots, x_n, y_n]$ be the polynomials such that*
$$\prod_{i=1}^{n}(x_i t + y_i) = F_0 t^n + F_1 t^{n-1} + \ldots + F_n \in A[t, x_1, y_1, \ldots, x_n, y_n].$$
Then for $S := \mathrm{Spec}(A)$ the morphism
$$\varphi_n \colon \mathbb{P}^1_S \times_S \ldots \times_S \mathbb{P}^1_S \to \mathbb{P}^n_S,\ ((x_1 : y_1), \ldots, (x_n : y_n)) \mapsto (F_0 : \ldots : F_n)$$
is finite surjective.

Proof. It suffices to show that φ is quasi-finite, which reduces us to the case that $A = k$ is a field. Take
$$z = (z_0 : \ldots : z_n) \in \mathbb{P}^n_S.$$
Note that $\varphi_n^{-1}(D^+(z_0))$ is the locus where each $x_i \neq 0$, $i = 1, \ldots, n$. The resulting morphism
$$\mathbb{A}^n_S \to \mathbb{A}^n_S,\ (y_1, \ldots, y_n) \mapsto (\sigma_1, \ldots, \sigma_n)$$
is given (up to sign) by the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_n$ in the y_i. This morphism is finite free of degree $n!$. Hence, we may assume that $z \in V^+(z_0)$. But $\varphi_n^{-1}(V^+(z_0))$ is
$$\coprod_{i=1}^{n} \mathbb{P}^1_S \times_S \ldots \times_S V^+(x_i) \times_S \ldots \mathbb{P}^1_S \to \mathbb{P}^{n-1}_S = V^+(z_0) \subseteq \mathbb{P}^n_S.$$
because $F_0 = x_1 \ldots x_n$. On each direct summand on the left identifies with φ_{n-1}. Hence, induction reduces to the case that $n = 1$. Here, the morphism is just the identity. □

Thus by 10.78 we can reduce to the case for $\mathbb{P}^1_S \to S$. In this case, we can apply 10.71 to see that it suffices to check that the following assertion.

Assume that A is a local henselian, noetherian ring, $S = \mathrm{Spec}(A)$ and $s \in S$ the unique closed point. Let $f \colon X \to S$ be a proper morphism, whose special fiber X_0 has dimension ≤ 1. Then for $n \geq 1$ the map
$$H^i_{\text{ét}}(X, \mathbb{Z}/n) \to H^i_{\text{ét}}(X_0, \mathbb{Z}/n)$$
is bijective for $i = 0$ and surjective for $i > 0$.

If $i = 0, 1$, then the assertion follows from 10.47 and 10.48.

Now, the case $i \geq 3$ is trivial by 10.39. Hence, assume $i = 2$. Clearly, one may assume that n is a prime power p^r. If $p = \mathrm{char}(k)$, where k is the residue field of A, then Artin-Schreiter theory implies $H^2_{\text{ét}}(X_0, \mathbb{Z}/p^r) = 0$, cf. 6.7. Thus, assume that n is invertible in k.

As X_0 has dimension ≤ 1, the map
$$\delta_{X_0} \colon \mathrm{Pic}(X_0) = H^1_{\text{ét}}(X_0, \mathbb{G}_m) \to H^2_{\text{ét}}(X_0, \underbrace{\mathbb{Z}/n(1)}_{\cong \mathbb{Z}/n})$$
is surjective (in fact this reduces to the case 9.5 by passing to normalizations and perfections as the Frobenius on $\mathrm{Pic}(X_0)$ induces multiplication by p and n is prime to p). By naturality of the connecting morphism $\delta_{(-)}$ it suffices to check the next theorem.

Theorem 10.80. *Let A be a local henselian ring, $S := \mathrm{Spec}(A)$ and $s \in S$ the unique closed point. Let $f \colon X \to S$ be a separated morphism of finite presentation and assume that the special fiber $X_0 = X \times_S s$ is of dimension ≤ 1. Then $\mathrm{Pic}(X) \to \mathrm{Pic}(X_0)$ is surjective.*

Proof. For simplicity, we assume that X is integral and that A is noetherian. The general case is proven in [13, Corollaire (21.9.12)]. As X_0 is quasi-projective (because X_0 is of dimension ≤ 1 and separated), each line bundle on X_0 can be represented by some Cartier divisor. Indeed, it is sufficient to construct a rational section of a line bundle and for this it suffices to see that the finite set of associated points of X_0 is contained in some affine open, cf. [13, Proposition 21.3.4]. But the last assertion is implied by quasi-projectivity of X_0. Hence, we have to see that the map $\mathrm{Div}(X) \to \mathrm{Div}(X_0)$ on Cartier divisors is surjective. Each Cartier divisor on X_0 is a linear combination of Cartier divisors, which are support at a single point in X_0 because X_0 is of dimension 1. Hence, let D_0 be an effective Cartier divisor with support a closed point $x_0 \in X_0$ and pick a section $t_0 \in \mathcal{O}_{X_0, x_0}$ such that $D_0 = V(t_0)$ in an open neighborhood of x_0 in X_0. On some affine open $U \subseteq X$ of x_0 we may find some $t \in \mathcal{O}_X(U)$ restricting to t_0. As we assume that X is integral the element t is regular on U. Set $Y := V(t) \subseteq U$. After shrinking U we may assume that $Y \cap X_0 = \{x_0\}$. Then $Y \to S$ is quasi-finite at x. By 10.16 we may write $Y = Y_1 \coprod Y_2$ with $Y_1 \to S$ *finite* and $Y_2 \cap X_0 = \emptyset$. Replacing U by its open subset $U \setminus Y_1$, we can assume that $Y \to S$ is finite. As $X \to S$ is separated, Y is therefore closed in X. Now, Y is an effective Cartier divisor on X, namely $X = U \cup X \setminus Y$ and Y is an effective Cartier divisor on Y and the empty divisor on $X \setminus Y$. This finishes the proof. □

At this point we have finished the proper base change theorem.[48]

11. Applications of the proper base change theorem

As in 4.22 the proper base change theorem implies that we can define a reasonable theory of exceptional pushforward, or cohomology with compact support.

More precisely we use Nagata's theorem on compactifications.

Theorem 11.1. *Let $f\colon X \to S$ be a separated morphism of finite type. Assume S is qcqs. Then there exists a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{j} & \overline{X} \\ & \searrow^{f} & \downarrow{\scriptstyle h} \\ & & S \end{array}$$

with j an open immersion and $h\colon \overline{X} \to S$ proper.

Proof. This is [Stacks, Tag 0F41]. □

Definition 11.2. Assume that $f\colon X \to S$ is separated of finite type and S qcqs. We define

$$Rf_!\colon \mathcal{D}^+_{\mathrm{tor}}(X) \to \mathcal{D}^+_{\mathrm{tor}}(S)$$

as $Rh_* \circ j_!$ for any compactification of f as in 11.1.

As the category of compactifications of $f\colon X \to S$ is cofiltered (by taking fiber products over S) this definition does not (up to isomorphism) depend on the compactification, cf. 4.21.

If $S = \mathrm{Spec}(k)$ with k a separably closed field, then we also write

$$R\Gamma_c(X, -) := Rf_!(-)$$

and

$$H^*_c(X, -)$$

for the "compactly supported cohomology".

The next lemma uses proper base change (and solves an exercise from 4.21).

Lemma 11.3. *Assume that $g\colon Y \to X, f\colon X \to S$ are separated and of finite finite type and that S is qcqs. Then there exists an isomorphism*

$$R(f\circ g)_! \cong Rf_! \circ Rg_!\colon \mathcal{D}^+_{\mathrm{tor}}(Y) \to \mathcal{D}^+_{\mathrm{tor}}(S)$$

of functors.

Proof. We can construct a diagram

$$\begin{array}{ccccc} Y & \xrightarrow{j_Y} & \overline{Y} & \xrightarrow{j'_X} & Z \\ & \searrow^{g} & \downarrow{\scriptstyle \overline{g}} & & \downarrow{\scriptstyle h} \\ & & X & \xrightarrow{j_X} & \overline{X} \\ & & & \searrow^{f} & \downarrow{\scriptstyle \overline{f}} \\ & & & & S \end{array}$$

with $\overline{f}, \overline{g}$ and h proper and j_Y, j_X open immersions and the top right square cartesion. Namely, start with a compactification $\overline{X}$ of X. Then let Z be a compactication of the morphism $Y \to X \to \overline{X}$ and define $\overline{Y} := X \times_{\overline{X}} Z$. The resulting morphism $Y \to \overline{Y}$ is then an open immersion as because the composition with the open immersion j'_X is an open immersion. Let $L \in \mathcal{D}^+_{\mathrm{tor}}(\overline{Y})$. Then the natural map

$$j_{X,!}R\overline{g}_*(L) \to Rh_*(j'_{X,!}(L))$$

is an isomorphism. Indeed, after applying j_X^* it is clearly an isomorphism (using that the diagram is cartesian and 5.40). After restricting to $\overline{X} \setminus X$ both sides are 0. For the left hand side this clear and for the right hand side this follows by the proper base change 10.1. If now $K \in \mathcal{D}^+_{\mathrm{tor}}(Y)$, then we can calculate

$$\begin{array}{rl} & Rf_! \circ Rg_!(K) \\ \cong & R\overline{f}_* j_{X,!} R\overline{g}_*(j_{Y,!}K) \\ \cong & R\overline{f}_* Rh_* j'_{X,!} j_{Y,!} K \\ \cong & R(\overline{f}\circ h)_*(j'_X \circ j_Y)_! K \\ \cong & R(f\circ g)_!(K) \end{array}$$

using that $Y \to Z$ is a compactification of Y over S. □

[48] Yeah!

Example 11.4. Let $f\colon X \to S$ be separated and of finite type with S qcqs.

(1) There exists a natural transformation $Rf_! \to Rf_*$, which is an isomorphism if f is proper. Indeed, this can be constructed using the natural transformation $j_! \to Rj_*$ for an open immersion.
(2) If f is étale, then using 11.5 and 10.9 one checks that $Rf_! \cong f_!$, where the LHS refers to 11.2 and the RHS to the $f_!$ used in 10.9.

As in 4.21 we get the following very useful version of proper base change.

Theorem 11.5. *Consider a cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of qcqs schemes with f separated and of finite type. Then there exists an isomorphism

$$g^* Rf_! \cong\cong Rf'_! g'^{,*}$$

of functors $\mathcal{D}^+_{\mathrm{tor}}(X) \to \mathcal{D}^+_{\mathrm{tor}}(S')$.

Proof. This reduces to the case that f is proper or an open immersion. The proper case is handled by 10.1 and the open immersion case was proven in 10.9. $\square$

If $g = \xi\colon S' = \mathrm{Spec}(\Omega) \to S$ for some separably closed field with image $s \in S$, then the stalk $Rf_!(K)_\xi$ can easily be calculated as $R\Gamma_c(X \times_S \mathrm{Spec}(\Omega), K_{|X \times_S \mathrm{Spec}(\Omega)})$. Now, $Rf_!$ can also be calculated locally on X, e.g., if $X = U \cup V$ is a union of two opens and $S = \mathrm{Spec}(k)$ with k a separably closed field, then there exists a distinguished triangle

$$R\Gamma_c(U \cap V, K_{|U \cap V}) \to R\Gamma_c(U, K_{|U}) \oplus R\Gamma_c(V, K_{|V}) \to R\Gamma_c(X, K)$$

for any $K \in \mathcal{D}^+_{\mathrm{tor}}(X)$. These two properties make $Rf_!$ much more accessible than Rf_*.

As an illustration let us prove the following theorem.

Theorem 11.6. *Let $f\colon X \to S$ be separated and of finite type with S qcqs and all fibers of dimension $\leq n$. Then*

$$R^i f_!(\mathcal{F}) = 0$$

for all $\mathcal{F} \in \mathrm{Sh}_{\mathrm{tor}}(X)$ and all $i > 2n$.

Proof. By 11.5 we may assume that $S = \mathrm{Spec}(k)$ with k separably closed. Then we can find an affine open $U \subseteq X$ such that $X \setminus U$ is of dimension $< n$. Using excision and induction this reduces to the case that $X = U$ is affine. Then we can embed $X \subseteq \mathbb{A}^m_k$ for some $m \gg 0$. Now, the projections

$$\mathbb{A}^m_k \to \mathbb{A}^{m-1}_k \to \ldots \to \mathbb{A}^1_k \to \mathrm{Spec}(k)$$

show that X can be written as an iterated relative curve. This reduces to the case that $n \leq 1$, where we proved the result in 10.39. $\square$

Theorem 11.7. *Let $f\colon X \to S$ be a separated morphism of finite type and S qcqs. If $\mathcal{F} \in \mathrm{Sh}_c(X_{\text{ét}})$ is constructible, then $R^i f_!(\mathcal{F})$ is constructible for each $i \geq 0$.*

Proof. This is [8, Arcata, IV.(6.2)]. Note that constructibility of $R^i f_*(\mathcal{F})$ can *not* be tested on geometric points of S. Hence, the argument proceeds by localizing on X and then reducing to the case that $f\colon X \to S$ is a proper, smooth curve. $\square$

Now assume that X is a scheme of locally finite type over $\mathrm{Spec}(\mathbb{C})$. By 5.4 we get a morphism of topoi

$$\alpha\colon \widetilde{X^{\mathrm{an}}} \cong \widetilde{X^{\mathrm{an}}_{\text{ét}}} \to \widetilde{X_{\text{ét}}}$$

of topoi such that $(-)^{\mathrm{an}} := \alpha^*$ sends an étale morphism $Y \to X$ to its analytification $f^{\mathrm{an}}\colon Y^{\mathrm{an}} \to X^{\mathrm{an}}$ from 3.24. The next theorem compares compactly supported étale cohomology of schemes with Betti cohomology.

Theorem 11.8. *Let $f\colon X \to S$ be a separated morphism of finite type of schemes of finite type over* $\mathrm{Spec}(\mathbb{C})$. *Then for any $K \in \mathcal{D}^+_{\mathrm{tor}}(X)$ there exists an isomorphism*

$$(Rf_!(K))^{\mathrm{an}} \to Rf^{\mathrm{an}}_!(K^{\mathrm{an}}).$$

In particular, $R\Gamma_c(X, K) \cong R\Gamma_c(X^{\mathrm{an}}, K^{\mathrm{an}})$.

Proof. Using the proper base change theorem on both sides reduces to the case that X is a proper, smooth curve over $S = \mathrm{Spec}(\mathbb{C})$ and $K = \mathbb{Z}/n$ for some $n \geq 1$. Then we can conclude by 6 and the GAGA theorem, 3.50. □

Finally, let us prove the important projection formula.

Theorem 11.9. *Let $f\colon X \to S$ be a separated morphism of finite type with S qcqs. Let Λ be a torsion ring. Then there exists an isomorphism of functors*

$$Rf_!(-) \otimes^L_{\underline{\Lambda}} (-) \cong Rf_!((-) \otimes^L_{\underline{\Lambda}} f^*(-))\colon D^+(X_{\text{ét}}, \underline{\Lambda}) \times D^+(S_{\text{ét}}, \underline{\Lambda}) \to D^+(S_{\text{ét}}, \underline{\Lambda}).$$

Proof. We may reduce to the case f is a quasi-compact open immersion or proper by using compatibility of both sides with compositions in f. The case that f is a quasi-compact open immersion can easily be checked on stalks. More generally, the case that f is qcqs étale can be reduced to checking that the natural morphism

$$f_!((-) \otimes^L_{\underline{X}} f^*(-)) \to f_!(-) \otimes^L_{\underline{\Lambda}} (-)$$

is an isomorphism, by checking this on stalks. If f is proper, then there exists a natural map

$$Rf_*(A) \otimes^L_{\underline{\Lambda}} B \to Rf_*(A \otimes_{\underline{\Lambda}} f^*B)$$

for $A \in D^+(X_{\text{ét}}, \underline{\Lambda})$ and $B \in D^+(S_{\text{ét}}, \underline{\Lambda})$. Now, one can reduce to the case that $B = g_!(\underline{\Lambda})$ for some qcqs étale map $g\colon U \to S$. Let $g'\colon X' := X \times_S U \to X$ and $f'\colon X' \to U$ be the projections. Now we calculate (using the projection formula for g, g' and proper base change)

$$\begin{array}{rl} & Rf_*(A) \otimes^L_{\underline{\Lambda}} g_!(\underline{\Lambda}) \\ \cong & g_!(g^*Rf_*(A)) \\ \cong & g_!(Rf'_*(g'^{,*}A)) \\ \cong & Rf_!(g'_!(g'^{,*}A)) \\ \cong & Rf_!(A \otimes^L_{\underline{\Lambda}} g'_!(\underline{\Lambda})) \\ \cong & Rf_!(A \otimes^L_{\underline{\Lambda}} f^*g_!(\underline{\Lambda})) \end{array}$$

as desired. □

12. 6-functor formalisms and Poincaré duality

This section is rather sketchy and the reader is advised to consult [30] for definite statements.

We assume that k is an algebraically closed field[49], and that all schemes in this section are separated and of finite type over $\mathrm{Spec}(k)$. We fix some $n \geq 1$, which is invertible in k and for a scheme X (subject to our conventions for this section) we set

$$D(X) := \mathcal{D}(X_{\text{ét}}, \underline{\mathbb{Z}/n}).$$

Under our restrictions, we get further cohomological vanishing results.

Theorem 12.1. *Assume that X is affine and $\mathcal{F} \in \mathrm{Sh}_{\mathrm{tor}}(X_{\text{ét}})$. Then*

$$H^i_{\text{ét}}(X, \mathcal{F}) = 0$$

for each $i > \dim(X)$.

Proof. This is [8, Arcata IV.(6.4)]. Let's do the reality check that X is a smooth curve and $\mathcal{F} = \underline{\mathbb{Z}/n}$. Let $j\colon X \to \overline{X}$ be the canonical compactification of X with non-empty boundary $i\colon Z \to \overline{X}$. We claim that $R^1j_*(\underline{\mathbb{Z}/n})$ is isomorphic to the skyscraper sheaf $i_*(\underline{\mathbb{Z}/n})$. This can be checked by passing to the strict henselizations, which are spectra discrete valuations rings (as $\overline{X}$ is a smooth curve over k). Let R be such a strict henselization with fraction field K, and let us identify $\underline{\mathbb{Z}/n} \cong \mu_n$. Then

$$H^1_{\text{ét}}(\mathrm{Spec}(K), \mu_n) \cong \mathbb{Z}/n$$

generated by the image of a uniformizer $\pi \in K^\times$ under the map $K^\times \to H^1_{\text{ét}}(\mathrm{Spec}(K), \mu_n)$ coming from Kummer theory. On the other hand, $H^1_{\text{ét}}(\mathrm{Spec}(R), \mu_n) = 0$ because R is strictly henselian. This shows the claim. Given that $R^1j_*(\underline{\mathbb{Z}/n}) \cong i_*(\mathbb{Z}/n)$ we consider the resulting distinguished triangle

$$\mathbb{Z}/n \to Rj_*(\mathbb{Z}/n) \to i_*(\mathbb{Z}/n)[-1],$$

which yields an exact sequence

$$\bigoplus_{z \in Z} \mathbb{Z}/n \cdot e_z \to \underbrace{H^2_{\text{ét}}(\overline{X}, \mathbb{Z}/n)}_{\cong \mathbb{Z}/n} \to H^2_{\text{ét}}(X, \mathbb{Z}/n).$$

[49] By 8.3 we could equivalently assume k separably closed as the base change to the algebraic closure would not change the étale site.

Now, $H^2_{\text{ét}}(\overline{X}, \mathbb{Z}/n) \cong \mathbb{Z}/n$ is generated by the Chern class of the line bundle $\mathcal{O}(x)$ for each point $x \in \overline{X}$ and one checks that each e_z mapsto the Chern class of $\mathcal{O}(z)$ (up to some sign). □

12.1 implies that for any $K \in D(X)$ the natural map

$$K \to R\lim_n \tau^{\geq -n} K$$

is an isomorphism, cf. [Stacks, Tag 0D6S]. This allows to extend the proper base change theorem and the projection formula to all of $D(X)$, and thus to define $Rf_!$ on all of $D(X)$.

Now, the association $X \mapsto D(X)$ is an example of a "6-functor formalism". This refers to the following data:

(1) for each X the tensor product functor $-\otimes_X - := -\otimes^L_{\mathbb{Z}/n} -\colon D(X) \times D(X) \to D(X)$, and the internal Hom-functor $R\mathcal{H}om_X(-,-)\colon D(X)^{\text{op}} \times D(X) \to D(X)$,
(2) for each $f\colon Y \to X$ a pullback functor $f^*\colon D(X) \to D(Y)$, which admits the right adjoint $Rf_*\colon D(Y) \to D(X)$,
(3) for each $f\colon Y \to X$ the "exceptional push forward" $Rf_!\colon D(Y) \to D(X)$, which admits a right adjoint $f^!\colon D(X) \to D(Y)$,

such that for $f\colon Y \to X$ the projection formula

$$Rf_!(-) \otimes_X (-) \cong Rf_!((-) \otimes_Y f^*(-))$$

is satisfied. Actually, Liu/Zhen and Mann have introduced a very precise notion of a 6-functor formalism, which is discussed nicely in Scholze's lectures on 6-functor formalisms in WS22/23 at the university of Bonn.

The existence of the "exceptional pullback" $f^!$ (which does not exist on the abelian level!) follows formally from the fact that $Rf_!$ commutes with all direct sums in $D(Y)$, cf. [Stacks, Tag 0A8G].

From now on we assume that the functors $Rf_!, Rf_*$ are derived, and therefore we drop the R.

Example 12.2. It follows formally from the theory of localization for topoi, cf. 5.25, and 11.4, that if $f\colon Y \to X$ is étale, then $f^! \cong f^*$.

We want to better understand the functor $f^!$. For X let $1_X \in D(X)$ be the unit for the tensor product, i.e., $1_X = \underline{\mathbb{Z}/n}$ is the constant sheaf associated with $\underline{\mathbb{Z}/n}$. Given a map $f\colon Y \to X$ and $A \in D(X)$ there exists a natural map

$$f^*(A) \otimes f^!(1_X) \to f^!(A).$$

Indeed, by adjunction we have to construct a map

$$f_!(f^*(A) \otimes f^!(1_X)) \overset{\text{projection formula}}{\cong} A \otimes f_!(f^!(1_X)) \to A$$

and here we can take the one induced by the counit $f_!(f^!(1_X)) \to 1_X$ (using that $1_X \otimes A \cong A$).

The object $f^!(1_X)$ is also called the dualizing complex for f.

Definition 12.3. We say that $f\colon Y \to X$ is weakly cohomologically smooth if

(1) for any $A \in D(X)$ the map $f^*(A) \otimes f^!(1_X) \to f^!(A)$, which constructed above, is an isomorphism.
(2) the object $f^!(1_X) \in D(Y)$ is invertible with respect to $\otimes$, and compatible with any base change, i.e, for any cartesian diagram

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

the natural map[50] $g'^{,*} f^!(1_X) \to f'^{,!}(1_{X'})$ is an isomorphism.

If any base change of $f\colon Y \to X$ along some $X' \to X$ is weakly cohomologically smooth, then f is called cohomologically smooth.

For example étale morphisms are cohomologically smooth (with $f^!(1_X) \cong \mathbb{Z}/n)$), and in 12.15 we will see that more generally any smooth morphism is cohomologically smooth.

Exercise 12.4. We leave it as an exercise that the class of cohomologically smooth morphisms is stable under base change and composition.

Let us first clarify how invertible objects in $D(X)$ look like.

[50] Adjoint to $f'_! g'^* f^!(1_X) \overset{\text{base change}}{\cong} g^* f_! f^!(1_X)) \to g^*(1_X) = 1_{X'}$ with the second map induced by the counit.

Lemma 12.5. *Let X be a scheme and assume that n is a prime power. Then $\mathcal{L} \in D(X)$ is invertible for $\otimes$ if and only if there exists an open closed decomposition $X = \coprod_{i\in I} X_i$ and $\mathbb{Z}/n$-local systems $\mathbb{L}_i$ of rank 1 on X_i such that $\mathcal{L}_{|X_i} \cong \mathbb{L}_i[n_i]$ for some $n_i \in \mathbb{Z}$.*

Proof. The proof can be found in [Stacks, Tag 0FPY]. □

It is a nice observation of Zavyalov [30] that checking cohomological smoothness is actually rather easy!

Let $f\colon Y \to X$ be a morphism, and let $p_1, p_2\colon Y \times_X Y \to Y$ be the projections, and $\Delta\colon Y \to Y \times_X Y$ the diagonal.

Definition 12.6. A trace-cycle theory on f is a triple $(\omega_f, \mathrm{tr}_f, \mathrm{cl}_\Delta)$ of

(1) an invertible object $\omega_f \in D(Y)$,
(2) a morphism $\mathrm{tr}_f\colon f_!\omega_f \to 1_X$, called “trace morphism”,
(3) a morphism $\mathrm{cl}_\Delta\colon \Delta_! 1_Y \to p_2^*\omega_f$, called “cycle map”,

such that the diagrams

$$\begin{array}{ccc} 1_Y & \xrightarrow{\cong} & p_{1,!}\Delta_!(1_Y) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle p_{1,!}(\mathrm{cl}_\Delta)} \\ 1_Y & \xleftarrow[\mathrm{tr}_{p_1}]{} & p_{1,!}(p_2^*\omega_f) \end{array}$$

and

$$\begin{array}{ccccc} \omega_f & \xrightarrow{\cong} & p_{2,!}(p_1^*\omega_f \otimes \Delta_!(1_Y)) & \xrightarrow{p_{2,!}(\mathrm{Id}\otimes \mathrm{cl}_\Delta)} & p_{2,!}(p_1^*\omega_f \otimes p_2^*\omega_f) \\ {\scriptstyle \cong}\downarrow & & & & \downarrow{\scriptstyle \cong} \\ \omega_f & \xleftarrow{\cong} & 1_X \otimes \omega_f & \xleftarrow{\mathrm{tr}_{p_2}\otimes \omega_f} & p_{2,!}p_1^*\omega_f \otimes \omega_f \end{array}$$

commute. Here, tr_{p_i} denotes the base change of tr_f. More precisely,

$$\mathrm{tr}_{p_1}\colon p_{1,!}(p_2^*(\omega_f)) \overset{\text{base change}}{\cong} f^*f_!(\omega_f) \overset{f^*(\mathrm{tr}_f)}{\to} f^*(1_X) = 1_Y$$

and similarly for tr_{p_2}.

Now, Zavyalov’s observation is the following.

Theorem 12.7 ([30, Theorem 3.3.1, Remark 3.3.2])**.** *A morphism $f\colon Y \to X$ is cohomologically smooth if and only if f admits a trace-cycle theory.*

Proof. Assume that f is cohomologically smooth. Then by definition $\omega_f := f^!(1_X)$ is invertible in $D(Y)$. Let $\mathrm{tr}_f\colon f_!(\omega_f) \to 1_X$ be the counit. Consider the cartesian diagram

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow{p_2} & Y \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{f} & X \end{array}$$

Then p_1 is cohomologically smooth by assumption, with dualizing complex

$$\omega_{p_1} = p_1^!(1_Y) \cong p_2^*(\omega_f).$$

This implies that $1_Y \cong \Delta^! p_1^!(1_Y) \cong \Delta^!(p_2^*(\omega_f))$, whose adjoint $\Delta_!(1_Y) \to p_2^*\omega_f$ we set as the cycle map cl_Δ. Now one checks that $(\omega_f, \mathrm{tr}_f, \mathrm{cl}_\Delta)$ is trace-cycle theory.

The converse in [30, Theorem 3.2.8] uses a bit more formalism, that we won’t develope here. In the end we need to see that the functor $f^*(-) \otimes \omega_f$ is adjoint to $f_!(-)$. Now, the map

$$f_!(f^*(-) \otimes \omega_f) \overset{\text{projection formula}}{\cong} (-) \otimes f_!(\omega_f) \overset{\mathrm{Id}\otimes \mathrm{tr}_f}{\to} (-)$$

will serve as the counit, and the unit will constructed using cl_Δ. The two diagrams in 12.6 the yield the triangle equalities needed for the adjunction. □

Let us construct the basic example of a trace-cycle theory.

Example 12.8. Consider $f\colon Y = \mathbb{P}^1_k \to \mathrm{Spec}(k)$ and assume that n is prime. By 12.5 and the fact that any finite, étale cover of $Y = \mathbb{P}^1_k$ splits, we see that the invertible objects in $D(Y)$ are exactly the $\mathbb{Z}/n[i]$ with $i \in \mathbb{Z}$. As f is proper, we have $f_! = f_*$. We claim that an interesting trace-cycle theory for f can only exist if $i = 2$ (and then it does exist!). Now, $f_!(\mathbb{Z}/n[i]) \cong \mathbb{Z}/n[i] \oplus \mathbb{Z}/n[i-2]$. Because n is prime, we thus have only the possibilities $i = 0, 2$ to construct some non-zero trace map

$$f_!(\mathbb{Z}/n[i]) \to \mathbb{Z}/n[0] = 1_{\mathrm{Spec}(k)}$$

and then we have (up to multiplying by some unit in $\mathbb{Z}/n$) take the respective projection. Let us analyze our possibilities for cycle class maps, i.e., maps $\Delta_! 1_Y = \Delta_* 1_Y \to p_2^*\mathbb{Z}/n[i] = \mathbb{Z}/n[i]$. Let $U := Y \times_{\mathrm{Spec}(k)} Y \setminus \Delta(Y)$ be the complement of the diagonal. Then we have a distinguished triangle

$$R\mathrm{Hom}_{Y\times Y}(i_*\mathbb{Z}/n, \mathbb{Z}/n[i]) \to R\mathrm{Hom}_{Y\times Y}(\mathbb{Z}/n, \mathbb{Z}/n[i]) \to R\mathrm{Hom}(j_!\mathbb{Z}/n, \mathbb{Z}/n[i]).$$

Now,

$$R\mathrm{Hom}_{Y\times Y}(\mathbb{Z}/n, \mathbb{Z}/n[i]) \cong \mathbb{Z}/n[i] \oplus \mathbb{Z}/n^{\oplus 2}[i+2] \oplus \mathbb{Z}/n[i+4]$$

by the Künneth formula (which is implied by the projection formula 11.9 here), and

$$R\mathrm{Hom}(j_!\mathbb{Z}/n, \mathbb{Z}/n[i]) \cong R\Gamma(U_{\text{ét}}, \mathbb{Z}/n[i]).$$

If $i = 0$, then we get a short exact sequence

$$\mathrm{Hom}_{Y\times Y}(\mathbb{Z}/n, \mathbb{Z}/n) \to \Gamma(Y \times Y, \mathbb{Z}/n) \overset{\cong}{\to} \Gamma(U, \mathbb{Z}/n),$$

which shows $\mathrm{Hom}_{Y\times Y}(\mathbb{Z}/n, \mathbb{Z}/n) = 0$. In particular, we can only expect some trace-cycle theory on Y for $i = 2$.

In order to finish the construction of a trace-cycle theory for $\mathbb{P}^1_k$ we introduce cohomology classes associated to divisors.

First we formalize the concept of an n-th root of a section a line bundle. For the next lemmata we can assume that X is a general scheme, i.e., not separated and of finite type over $\mathrm{Spec}(k)$. Let $m \geq 1$.

Lemma 12.9. *Let X be scheme, $\mathcal{L}$ a line bundle on X and $s \in \Gamma(X, \mathcal{L}^{\otimes m})$. Then the functor*

$$F\colon (\mathrm{Sch}/X)^{\mathrm{op}} \to (\mathrm{Sets}),\ (T \overset{f}{\to} X) \mapsto \{x \in f^*(\mathcal{L}) \mid x^m = s \in \mathcal{L}^{\otimes m}\}$$

is representable by some finite, locally free scheme $Z_{\mathcal{L},s}$ over X with a μ_m-action, which makes $Z_{\mathcal{L},s}$ into a μ_m-torsor if (and only if) $s\colon \mathcal{O}_X \to \mathcal{L}^{\otimes m}$ is an isomorphism.

If m is invertible on X and s an isomorphism, then the μ_m-torsor $Z_{\mathcal{L},s}$ is finite, étale and hence defines a class $\mathrm{cl}(Z_{\mathcal{L},s}) \in H^1_{\text{ét}}(X, \mu_m)$.

Proof. First note that μ_m acts on the functor F, via

$$\mu_m(T) \times F(\mu_m) \to F(\mu_m),\ (\zeta, x) \mapsto \zeta \cdot x.$$

Given this, all the claims are local on X and hence we way assume that $\mathcal{L} \cong \mathcal{O}_X$ and $X = \mathrm{Spec}(A)$ is affine. Then F is represented by $\mathrm{Spec}(A[T]/T^m - s)$, which is finite, free over X. If $s \in A$ is invertible, then after replacing A by any faithfully flat A-algebra B such that s admits an m-th root, e.g., $B = A[T]/(T^m - s)$, we may assume that there exists some $x \in A^\times$ such that $x^m = s$. Then

$$A[T]/(T^m - s) \cong A[\tilde{T}]/(\tilde{T}^m - 1),\ T \mapsto \tilde{T}/x,$$

μ_n-equivariantly. Now, $\mathrm{Spec}(A[\tilde{T}]/(\tilde{T}^m - 1)) \cong \mu_m$ over $\mathrm{Spec}(A)$, which finishes the proof. □

We now construct classes associated with divisors.

Definition 12.10. Let X be a scheme and $i\colon D \to X$ the inclusion of an effective Cartier divisor with complement $j\colon U \to X$.

(1) For a sheaf $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(X_{\text{ét}})$ of $\mathbb{Z}/m$-modules we set $H^j_D(X_{\text{ét}}, \mathcal{F}) := \mathrm{Hom}_X(i_*(\mathbb{Z}/m), \mathcal{F}[j])$ and call it the cohomology of $\mathcal{F}$ with support in D.
(2) Let $s\colon \mathcal{O}_X \to \mathcal{O}_X(D)$ be the section defining D. Then we set

$$\mathrm{cl}(D) \in H^2_D(X_{\text{ét}}, \mu_m)$$

as the image of $\mathrm{cl}(Z_{\mathcal{O}_X(D)_{|U}, s_{|U}}) \in H^1_{\text{ét}}(U, \mu_m)$.

Note that the distinguished triangle $0 \to j_!\mathbb{Z}/m \to \mathbb{Z}/m \to i_*\mathbb{Z}/m \to 0$ yields a distinguished triangle

$$R\mathrm{Hom}_X(i_*(\mathbb{Z}/m), \mathcal{F}) \to R\mathrm{Hom}_X(\mathbb{Z}/m, \mathcal{F}) \to R\mathrm{Hom}_X(j_!(\mathbb{Z}/m), \mathcal{F}),$$

which in turn yields a long exact sequence

$$\ldots \to H^j_{\text{ét}}(U, \mathcal{F}) \to H^{j+1}_D(X, \mathcal{F}) \to H^{j+1}_{\text{ét}}(X, \mathcal{F}) \to H^{j+1}_{\text{ét}}(U, j^*\mathcal{F}) \to \ldots$$

because $R\mathrm{Hom}_X(j_!\mathbb{Z}/m, \mathcal{F}) \cong R\mathrm{Hom}_U(\mathbb{Z}/m, j^*\mathcal{F}) \cong R\Gamma_{\text{ét}}(U, j^*\mathcal{F})$. This defines the natural map alluded to in 12.10.

Remark 12.11. The image of $\mathrm{cl}(D) \in H^2_D(X, \mu_m)$ in $H^2(X, \mu_m)$ under the natural map (induced by $\mathbb{Z}/m \to i_*(\mathbb{Z}/m)$) is by construction the first Chern class $c_1(\mathcal{O}_X(D))$.

Remark 12.12. If $f\colon Y \to X$ is a morphism, such that $D \times_X Y$ is again an effective Cartier divisors, then $f^*(\mathrm{cl}(D)) = \mathrm{cl}(D \times_X Y)$. Thus, cycle classes of effective Cartier divisors are natural.

Now, let us come back to 12.8 and in particular again assume that all schemes are separated and of finite type over $\mathrm{Spec}(k)$.

Example 12.13. We continue with 12.8 and fix an isomorphism $\mu_n \cong \mathbb{Z}/n$. We have an exact sequence

$$\underbrace{H^1_{\text{ét}}(Y \times Y, \mathbb{Z}/n)}_{=0} \to H^1_{\text{ét}}(U, \mathbb{Z}/n) \to H^2_{\Delta}(Y \times Y, \mathbb{Z}/n) \to H^2_{\text{ét}}(Y \times Y, \mathbb{Z}/n) \xrightarrow{\alpha} H^2_{\text{ét}}(U, \mathbb{Z}/n)$$

and we can use this to construct interesting morphisms $i_*(\mathbb{Z}/n) \to \mathbb{Z}/n[2])$. By the Künneth formula $H^2_{\text{ét}}(Y \times Y, \mathbb{Z}/n) = \mathbb{Z}/n \cdot \mathrm{cl}(Y \times \{\infty\}) \oplus \mathbb{Z}/n \cdot \mathrm{cl}(\{\infty\} \times Y)$. We claim that α is injective. For this we consider the composition

$$\alpha_1\colon H^2_{\text{ét}}(Y \times Y, \mathbb{Z}/n) \to H^2_{\text{ét}}(U, \mathbb{Z}/n) \to H^2_{\text{ét}}(U \setminus \{\infty\} \times Y, \mathbb{Z}/n),$$

which kills the class $\mathrm{cl}(\{\infty\} \times Y)$ (by naturality of the cycle classes, 12.12). On the other hand $\alpha_1(\mathrm{cl}(Y \times \{\infty\}))$ restricts to $\mathrm{cl}(\{0\} \times \{\infty\})$ on $U \cap (\{0\} \times Y) \cong \mathbb{P}^1 \setminus \{0\}$ and this class in

$$H^2_{\{\infty\}}(\mathbb{P}^1_k \setminus \{0\}, \mathbb{Z}/n) \cong H^1_{\text{ét}}(\mathbb{P}^1_k \setminus \{0, \infty\}, \mathbb{Z}/n) \cong \mathbb{Z}/n$$

is the class of the divisor ∞ in $\mathbb{P}^1_k \setminus \{0\}$, which corresponds to the μ_n-torsor of n-th roots of T on $\mathbb{P}^1_k \setminus \{0, \infty\} \cong \mathrm{Spec}(k[T, T^{-1}])$. Here, we used $H^1_{\text{ét}}(\mathbb{A}^1_k, \mathbb{Z}/n) \cong H^2_{\text{ét}}(\mathbb{A}^1_k, \mathbb{Z}/n) = 0$, cf. 9.6, to get $H^2_{\{\infty\}}(\mathbb{P}^1_k \setminus \{0\}, \mathbb{Z}/n) \cong H^1_{\text{ét}}(\mathbb{P}^1_k \setminus \{0, \infty\}, \mathbb{Z}/n)$. Using now the map

$$\alpha_2\colon H^2_{\text{ét}}(Y \times Y, \mathbb{Z}/n) \to H^2_{\text{ét}}(U, \mathbb{Z}/n) \to H^2_{\text{ét}}(U \setminus Y \times \{\infty\}, \mathbb{Z}/n)$$

with an analogous calculation, we see that α is injective. As a consequence of the injectivity of α we can conclude that (canonically)

$$\mathrm{Hom}_{Y \times Y}(\Delta_*(\mathbb{Z}/n), \mu_n[2]) \cong H^2_{\Delta}(Y \times Y, \mu_n) \cong H^1_{\text{ét}}(U, \mu_n) \cong \mathbb{Z}/n,$$

generated by the class $\mathrm{cl}(\Delta)$.

We can now construct the trace-cycle theory for $\mathbb{P}^1_k$.

Lemma 12.14. *Let $f\colon Y := \mathbb{P}^1_k \to X := \mathrm{Spec}(k)$ and set $\omega_f := \mu_n[2]$. Let*

$$\mathrm{tr}_f\colon Rf_*(\omega_f) \to 1_X$$

be the unique morphism mapping the first Chern class $c_1(\mathcal{O}_{\mathbb{P}^1_k}(1)) \in H^2(\mathbb{P}^1_k, \mu_n) = \mathcal{H}^0(Rf_*(\omega_f))$ to $1 \in \mathbb{Z}/n = \mathcal{H}^0(1_X)$. Let

$$\mathrm{cl}_\Delta\colon \Delta_*(1_Y) \to p_2^*(\omega_f) = \mu_n[2]$$

be $\mathrm{cl}(\Delta) \in H^2_{\Delta}(Y \times Y, \mu_n) \cong \mathrm{Hom}(\Delta_*(\mathbb{Z}/n), \mu_n[2])$. Then $(\omega_f, \mathrm{tr}_f, \mathrm{cl}_\Delta)$ is a trace-cycle theory for $\mathbb{P}^1_k$.

Proof. First note that $\mathrm{tr}_{p_1}, \mathrm{tr}_{p_2}$ are similarly the unique maps such that the first Chern classes of $\mathcal{O}_{Y \times Y}(Y \times \{\infty\})$ resp. $\mathcal{O}_{Y \times Y}(\{\infty\} \times Y)$ map to the element $1 \in \mathbb{Z}/n$. The commutativity of

$$\begin{array}{ccc} 1_Y & \xrightarrow{\cong} & p_{1,!}\Delta_!(1_Y) \\ \downarrow{=} & & \downarrow{p_{1,!}(\mathrm{cl}_\Delta)} \\ 1_Y & \xleftarrow[\mathrm{tr}_{p_1}]{} & p_{1,!}(p_2^*\omega_f) \end{array}$$

can be checked after base change to closed points on Y. Let $y \in \mathbb{P}^1_k$ be closed, then composition on the right identifies with the map

$$\mathbb{Z}/n \stackrel{\mathrm{cl}(\{y\})}{\to} H^2_{\{y\}}(\mathbb{P}^1_k, \mu_n) \to H^2(\mathbb{P}^1_k, \mu_n) \stackrel{\mathrm{tr}_f}{\to} \mathbb{Z}/n,$$

and this is the identity because $\mathcal{O}_Y(y) \cong \mathcal{O}_Y(1)$. The commutativity of

$$\begin{array}{ccccc} \omega_f & \xrightarrow{\cong} & p_{2,!}(p_1^*\omega_f \otimes \Delta_!(1_Y)) & \xrightarrow{p_{2,!}(\mathrm{Id}\otimes\mathrm{cl}_\Delta)} & p_{2,!}(p_1^*\omega_f \otimes p_2^*\omega_f) \\ \downarrow{\cong} & & & & \downarrow{\cong} \\ \omega_f & \xleftarrow{\cong} & 1_X \otimes \omega_f & \xleftarrow{\mathrm{tr}_{p_2}\otimes\omega_f} & p_{2,!}p_1^*\omega_f \otimes \omega_f \end{array}$$

can be checked after restricting to the fiber of p_2 over some $y \in \mathbb{P}^1_k$ closed. Then the composition along the right side of the diagram identifies with the map

$$\mu_n[2] \cong R\Gamma(\mathbb{P}^1_k, \mu_n[2] \otimes i_{y,*}(\mathbb{Z}/n)) \to R\Gamma(\mathbb{P}^1_k, \mu_n[2] \otimes \mu_n[2])) \to R\Gamma(\mathbb{P}^1_k, \mu_n[2]) \otimes \mu_n[2] \stackrel{\mathrm{tr}_f}{\to} \mathbb{Z}/n \otimes \mu_n[2],$$

which is the identity because the class of y mapsto $1 \in \mathbb{Z}/n$ under tr_f. $\square$

For $d \in \mathbb{Z}$ and a scheme X we define the d-th Tate twist $1_X(d) := \mathbb{Z}/n(1)^{\otimes d}$, where $\mathbb{Z}/n(1) = \mu_n$ denotes the étale sheaf of n-th roots of unity (which secretely is isomorphic to $\underline{\mathbb{Z}/n}$ after choosing a primitive n-th root of unity in k[51]).

We can deduce the following.

Theorem 12.15. *If $f\colon Y \to X$ is smooth, then f is cohomologically smooth and, if f has constant relative dimension d, then the dualizing complex ω_f is naturally isomorphic to $\mu_n^{\otimes d}[2d]$.*

Proof. Note that 12.14, 12.7, 12.4 and the fact that étale morphisms are cohomologically smooth, that every scheme étale over some relative $\mathbb{A}^n_X$ is cohomologically smooth over X. Using descent of ∞-categories, one checks that the property of being cohomologically smooth is local on Y, cf. [30, Lemma 2.3.16]. The identification of ω_f is discussed in [30, Section 4]. This finishes a sketch of an argument. □

We can deduce now easily that Poincaré duality holds.

Example 12.16. Assume that $f\colon X \to S := \mathrm{Spec}(k)$ is proper and cohomologically smooth. In particular, $Rf_* \cong Rf_!$ and for $A \in D(X), B \in D(\mathrm{Spec}(k)) \cong D(\mathbb{Z}/n)$ there exists an isomorphism[52]

$$R\mathrm{Hom}_S(Rf_*(A), B) \cong R\mathrm{Hom}_X(A, f^*(B) \otimes f^!(\mathbb{Z}/n)).$$

Now, assume that $A \cong \mathbb{Z}/n, B \cong \mathbb{Z}/n$. Using $f^!(\mathbb{Z}/n) \cong \mathbb{Z}/n(d)[2d] := \mu_n^{\otimes d}[2d]$ we obtain

$$R\mathrm{Hom}_{\mathbb{Z}/n}(R\Gamma(X_{\text{ét}}, \mathbb{Z}/n), \mathbb{Z}/n) \cong R\mathrm{Hom}_X(\mathbb{Z}/n, f^*(\mathbb{Z}/n) \otimes \mathbb{Z}/n(d)[2d]) \cong R\Gamma(X_{\text{ét}}, \mathbb{Z}/n(d)[2d]).$$

Now we look at the i-th cohomology object for some $i \in \mathbb{Z}$. Using that $\mathrm{Hom}_{\mathbb{Z}/n}(-, \mathbb{Z}/n)$ is an exact functor ($\mathbb{Z}/n$ is injective as an $\mathbb{Z}/n$-module by Baer's theorem), we get

$$\begin{array}{rl} & \mathrm{Hom}_{\mathbb{Z}/n}(H^{-i}_{\text{ét}}(X, \mathbb{Z}/n), \mathbb{Z}/n) \\ \cong & \mathrm{Hom}_{\mathbb{Z}/n}(\mathcal{H}^{-i}(R\Gamma(X_{\text{ét}}, \mathbb{Z}/n)), \mathbb{Z}/n) \\ \cong & \mathcal{H}^i(R\Gamma(X_{\text{ét}}, \mathbb{Z}/n(d)[2d]) \\ \cong & H^{2d+i}_{\text{ét}}(X, \mathbb{Z}/n(d)). \end{array}$$

Taking $i \in [-2d, 0]$ shows that f satisfies Poincaré duality if it satisfies a much stronger form of the usual Poincaré duality for compact (real or complex) manifolds.

In particular if $S = \mathrm{Spec}(k)$, then

$$\mathrm{Hom}_{\mathbb{Z}/n}(H^i_c(X, \mathbb{Z}/n), \mathbb{Z}/n) \cong H^{2d-i}(X, \mathbb{Z}/n(d)).$$

for $d := \dim(X)$. This allows us to compute the cohomology of $\mathbb{A}^d_k$.

Example 12.17. We have

$$H^i_{\text{ét}}(\mathbb{A}^d_k, \mathbb{Z}/n) \cong \begin{cases} \mathbb{Z}/n, & i = 0 \\ 0, & i > 0. \end{cases}$$

Indeed, we know $R\Gamma_c(\mathbb{A}^d_k, \mathbb{Z}/n) \cong \mathbb{Z}/n(d)[2d]$ (by induction and embedding into $\mathbb{P}^d_k$ for example). Now we can use 12.16.

For further applications of Poincaré duality we refer to [30, Section 1.3].

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